Chapter 5

On Hamming and b-symbol Distances of Constacyclic Codes of Length $4p^s$ over $\Re$

5.1 Introduction

Let $\mathbb{F}_{p^m}$ be the finite field of order p^m, where p is an odd prime, and m is a positive integers such that $p^m \equiv 1 \pmod 4$ holds. Here, we determine the Hamming and b-symbol distances for all Λ-constacyclic codes of length $4p^s$ over $\Re$ for any non-square unit Λ and positive integer s, and use this distance distribution to provide several codes with new parameters. As an application, we identify all the MDS codes among such codes with respect to the Hamming and b-symbol distances.

The rest of the Chapter is organized as follows. In Section 5.2, we obtain the Hamming distances of all constacyclic codes of length $4p^s$ over $\Re$ and provide some ex-

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amples. In Section 5.3, we identify all MDS constacyclic codes of length $4p^s$ over $\Re$ with respect to the Hamming distance. In Section 5.4, we determine the b-symbol distances for all b-symbol constacyclic codes of length $4p^s$ over $\Re$. And in Section 5.5, we determine all MDS b-symbol constacyclic codes of length $4p^s$ over $\Re$.

5.2 Hamming Distance of Codes over $\Re$

In [3], Dinh et al. determined the structures of all Λ-constacyclic codes of length $4p^s$ over $\Re$ for $p^m \equiv 1 \pmod 4$, where Λ is a non-square unit of $\Re$ as follows

Theorem 5.2.1. *[3] Let $\Lambda \in \Re$ be a non-square unit of $\Re$.*

1. *If Λ is of form $(\Phi + u\Psi)$, then the ring $\Re_{\Phi+u\Psi} = \frac{\Re[x]}{\langle x^{4p^s} - (\Phi+u\Psi)\rangle}$ is a finite chain ring having maximal ideal $\langle (x^4 - \Phi_0)\rangle$, where $\Phi_0 \in \mathbb{F}_{p^m}$, satisfying $\Phi = {\Phi_0}^{p^s}$. Also, in $\Re_{\Phi+u\Psi}$, we have $\langle (x^4 - \Phi_0)^{p^s}\rangle = \langle u\rangle$. Therefore, $(\Phi + u\Psi)$-constacyclic codes of length $4p^s$ over $\Re$ are the ideals $\langle (x^4 - \Phi_0)^{\jmath}\rangle$, $0 \le \jmath \le 2p^s$ of the ring $\Re_{\Phi+u\Psi} = \frac{\Re[x]}{\langle x^{4p^s} - (\Phi+u\Psi)\rangle}$. Also, the number of codewords is $|C| = p^{4m(2p^s - \jmath)}$.*

2. *If Λ is a unit of form Ω, then the ring $\Re_\Omega = \frac{\Re[x]}{\langle x^{4p^s} - \Omega\rangle}$ is a local finite non chain ring with the non principal maximal ideal $\langle u, x^4 - \Omega_0\rangle$, where $\Omega_0 \in \mathbb{F}_{p^m}$ satisfying $\Omega = {\Omega_0}^{p^s}$. Therefore, Ω-constacyclic codes of length $4p^s$ over $\Re$ are the ideals of the ring $\Re_\Omega = \frac{\Re[x]}{\langle x^{4p^s} - \Omega\rangle}$, and are given by:*

    - *Type 1: $\langle 0\rangle$, $\langle 1\rangle$, which are trivial ideals. The number of codewords are 1 and p^{8mp^s} respectively.*
    - *Type 2: $\langle u(x^4 - \Omega_0)^{\jmath}\rangle$, where $0 \le \jmath \le p^s - 1$ i.e., Type 2 consists non-monic polynomial generated by principal ideals. Also, the number of codewords is $|C| = p^{4m(p^s - \jmath)}$.*

- *Type* 3*:* $\langle (x^4-\Omega_0)^{\jmath} + u(x^4-\Omega_0)^{\ell}\Im(x)\rangle$*, where* $1 \le \jmath \le p^s-1$*,* $0 \le \ell < \jmath$*, and either* $\Im(x)$ *is a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\Omega\rangle}$ *or* 0 *i.e., Type* 3 *consists monic polynomial generated by principal ideals. Also, the number of codewords is* $|C| = p^{8m(p^s-\jmath)}$*, when* $\Im(x)=0$ *and*

$$|C| = \begin{cases} p^{8m(p^s-\jmath)}, & \textit{If } 1 \le \jmath \le \frac{p^s+\ell}{2} \\ p^{4m(p^s-\ell)}, & \textit{If } \frac{p^s+\ell}{2} < \jmath \le p^s-1, \end{cases}$$

  when $\Im(x) \neq 0$*.*

- *Type* 4*:* $\langle (x^4-\Omega_0)^{\jmath} + u(x^4-\Omega_0)^{\ell}\Im(x), u(x^4-\Omega_0)^{\zeta}\rangle$*, where* $\Im(x)$ *is the same as in Type* 3*,* $\deg(\Im) \le \zeta-\ell-1$*,* $\zeta < \chi$*, and* χ *is the smallest possible integer such that* $u(x^4-\Omega_0)^{\chi} \in \langle (x^4-\Omega_0)^{\jmath} + u(x^4-\Omega_0)^{\ell}\Im(x)\rangle$*; i.e.,* $\chi=\jmath$*, if* $\Im(x)=0$*, otherwise* $\chi = \min\{\jmath, p^s-\jmath+\ell\}$ *i.e. non-principal ideals. Also, the number of codewords is* $|C| = p^{4m(2p^s-\jmath-\zeta)}$*.*

Proposition 5.2.2. *[3] Let* χ *be the smallest integer such that* $u(x^4-\Omega_0)^{\chi} \in \langle (x^4-\Omega_0)^{\jmath} + u(x^4-\Omega_0)^{\ell}\Im(x)\rangle$*. Then*

$$\chi = \begin{cases} \jmath, & \textit{if } \Im(x)=0 \\ \min\{\jmath, p^s-\jmath+\ell\}, & \textit{if } \Im(x)\neq 0 \end{cases}$$

In [1, 4], the Hamming distances of all λ-constacyclic codes over $\mathbb{F}_{p^m}$ of length $4p^s$, where λ is a non-square unit and $p^m \equiv 1 \pmod 4$, are completely determined. We list this result in its simple form as follows:

Theorem 5.2.3. *[1, 4] Let* $C = \langle (x^4-\Phi_0)^{\jmath}\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\Phi\rangle}$*, for* $0 \le \jmath \le p^s$*, then the*

Hamming distance distribution $d_H(C)$ *is completely given by*

$$d_H(C)=\begin{cases}1, & \textit{if } \jmath=0\\ (\wp+1)p^{\sigma}, & \textit{if } p^s-p^{s-\sigma}+(\wp-1)p^{s-\sigma-1}\\ & +1\leq \jmath\leq p^s-p^{s-\sigma}+\wp p^{s-\sigma-1}\\ 0, & \textit{if } \jmath=p^s\end{cases}$$

where $1\leq\wp\leq p-1$, $0\leq\sigma\leq s-1$.

When $\Lambda=\Phi+u\Psi$, then the Hamming distance distribution $d_H(C)$ of $(\Phi+u\Psi)$-constacyclic codes of length $4p^s$ over $\Re$ is completely determined as follows:

Theorem 5.2.4. *Let* $C\subseteq\Re_{\Phi+u\Psi}=\frac{\Re[x]}{\langle x^{4p^s}-(\Phi+u\Psi)\rangle}$, *then* $C=\langle(x^4-\Phi_0)^{\jmath}\rangle$, *for* $\jmath\in\{0,1,\dots,2p^s\}$. *Then the Hamming distance distribution* $d_H(C)$ *is given by*

$$d_H(C)=\begin{cases}1, & \textit{if } 0\leq\jmath\leq p^s\\ (\wp+1)p^{\sigma}, & \textit{if } 2p^s-p^{s-\sigma}+(\wp-1)p^{s-\sigma-1}\\ & +1\leq \jmath\leq 2p^s-p^{s-\sigma}+\wp p^{s-\sigma-1}\\ 0, & \textit{if } \jmath=2p^s\end{cases}$$

where $1\leq\wp\leq p-1$, $0\leq\sigma\leq s-1$.

Proof. We consider three cases.

Case 1: If $\jmath=0$ and $\jmath=2p^s$, then we have trivial constacyclic codes $\langle 1\rangle$ and $\langle 0\rangle$ having $d_H(C)=1$ and 0, respectively.

Case 2: $1\leq\jmath\leq p^s$. The ring $\Re_{\Phi+u\Psi}$ is a chain ring whose ideals are $\Re_{\Phi+u\Psi}=\langle 1\rangle\supsetneq\langle(x^4-\Phi_0)\rangle\supsetneq\cdots\supsetneq\langle(x^4-\Phi_0)^{p^s}\rangle\supsetneq\cdots\supsetneq\langle(x^4-\Phi_0)^{2p^s}\rangle=\langle 0\rangle$. So, clearly we have $u\in\langle(x^4-\Phi_0)^{\jmath}\rangle$ and hence we have Hamming distance, $d_H(C)=1$.

Case 3: $p^s + 1 \leq \jmath \leq 2p^s - 1$. Then, we have $\langle (x^4 - \Phi_0)^\jmath \rangle = \langle u(x^4 - \Phi_0)^{\jmath - p^s} \rangle$. So, obviously the codewords of the code $\langle (x^4 - \Phi_0)^\jmath \rangle$ in $\Re_{\Phi + u\Psi}$ are exactly the same as the codewords of the code $\langle (x^4 - \Phi_0)^{\jmath - p^s} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Phi \rangle}$ multiplied by u, having the same Hamming weights. Moreover, the codes $\langle (x^4 - \Phi_0)^{\jmath - p^s} \rangle$ of length $4p^s$ have Hamming distances determined in Theorem 5.2.3. $\square$

Now, we are going to determine the Hamming distance distribution for the rest of Λ-constacyclic codes, where $\Lambda = \Omega \in \mathbb{F}_{p^m}^*$. Obviously, we know that $\mathbb{F}_{p^m}$ is a subring of $\Re$. From now, we will denote $d_H(C_F)$ as the Hamming distance of the code C over $\mathbb{F}_{p^m}$.

For each codeword $\mathbf{c} = (c_0, c_1, c_2, \ldots, c_{n-1})$ over $\Re$, the polynomial representation of $c(x)$ is given by $c(x) = \tilde{a}(x) + u\tilde{b}(x)$, where $\tilde{a}(x), \tilde{b}(x)$ are polynomials over $\mathbb{F}_{p^m}$, with corresponding codewords $\tilde{\mathbf{a}} = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{n-1})$ and $\tilde{\mathbf{b}} = (\tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_{n-1})$, respectively, over $\mathbb{F}_{p^m}$. As $c_i = \tilde{a}_i + u\tilde{b}_i$, $c_i = 0$ if and only if $\tilde{a}_i = \tilde{b}_i = 0$, hence we can conclude that $\mathrm{wt}_H(c(x)) \geq \max\{\mathrm{wt}_H(\tilde{a}(x)), \mathrm{wt}_H(\tilde{b}(x))\}$.

Now, we determine the Hamming distance for each type of Ω-constacyclic codes of length $4p^s$ one by one.

Obviously, the trivial ideals $\langle 0 \rangle$, $\langle 1 \rangle$ reside in Type 1, and hence, their Hamming distances are given by 0 and 1, respectively.

The Hamming distance of Type 2 Ω-constacyclic codes can be determined as follows:

Theorem 5.2.5. *Let $C = \langle u(x^4 - \Omega_0)^\jmath \rangle$, $0 \leq \jmath \leq p^s - 1$ be a Ω-constacyclic code of Type 2 of length $4p^s$ over $\Re$. Then, we have $d_H(C) = d_H(\langle (x^4 - \Omega_0)^\jmath \rangle_F)$, and $d_H(C)$*

is given by

$$d_H(C) = \begin{cases} 1, & \text{if } \jmath = 0 \\ (\wp+1)p^{\sigma}, & \text{if } p^s - p^{s-\sigma} + (\wp-1)p^{s-\sigma-1} \\ & +1 \leq \jmath \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1} \end{cases}$$

where $1 \leq \wp \leq p-1$*,* $0 \leq \sigma \leq s-1$*.*

Proof. We consider the following two cases

Case 1: If $\jmath = 0$, then trivially we have, $d_H(C) = 1$.

Case 2: If $p^s - p^{s-\sigma} + (\wp-1)p^{s-\sigma-1} + 1 \leq \jmath \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then for a Type 2 code $C = \langle u(x^4 - \Omega_0)^{\jmath} \rangle$, $0 \leq \jmath \leq p^s - 1$, the codewords of the code C are exactly same as the codewords of the Ω-constacyclic codes $\langle (x^4 - \Omega_0)^{\jmath} \rangle$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Omega \rangle}$ multiplied by u. Thus, we get $d_H(C) = d_H(\langle (x^4 - \Omega_0)^{\jmath} \rangle_F)$ and are given by Theorem 5.2.3. $\square$

Now, we discuss the Hamming distance of Type 3 Ω-constacyclic codes of length $4p^s$.

Theorem 5.2.6. *Let* $C = \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell} \Im(x) \rangle$ *be a* Ω*-constacyclic code of Type* 3 *of length* $4p^s$ *over* $\Re$*, where* $1 \leq \jmath \leq p^s - 1$*,* $0 \leq \ell < \jmath$ *and either* $\Im(x)$ *is a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Omega \rangle}$ *or* 0*. Then, we have* $d_H(C) = d_H(\langle (x^4 - \Omega_0)^{\chi} \rangle_F)$*, where* χ *is the smallest integer such that* $u(x^4 - \Omega_0)^{\chi} \in \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell} \Im(x) \rangle$*, which is given by*

$$\chi = \begin{cases} \jmath, & \text{if } \Im(x) = 0 \\ \min\{\jmath, p^s - \jmath + \ell\}, & \text{if } \Im(x) \neq 0 \end{cases}$$

and is determined by

$$d_H(C) = (\wp+1)p^{\sigma},$$

where $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, $1 \leq \wp \leq p-1$ *and* $0 \leq \sigma \leq s-1$.

Proof. Since χ is the smallest integer such that $u(x^4-\Omega_0)^\chi \in \langle (x^4-\Omega_0)^\jmath + u(x^4-\Omega_0)^\ell \Im(x)\rangle$, we have,

$$d_H(C) \leq d_H(\langle u(x^4-\Omega_0)^\chi\rangle) = d_H(\langle (x^4-\Omega_0)^\chi\rangle_F).$$

Now, let us take an arbitrary polynomial $c(x) \in C$. So, there exist two polynomials, $f_0(x)$ and $f_u(x)$ over $\mathbb{F}_{p^m}$ such that

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(x^4-\Omega_0)^\jmath + u(x^4-\Omega_0)^\ell \Im(x)] \\ &= f_0(x)(x^4-\Omega_0)^\jmath + u[f_0(x)(x^4-\Omega_0)^\ell \Im(x) \\ &\quad + f_u(x)(x^4-\Omega_0)^\jmath]. \end{aligned}$$

Now, we consider two cases as follows:

Case 1: When $\Im(x) = 0$, then we have

$$\begin{aligned} &\mathrm{wt}_H(c(x)) \\ &\geq \max\left\{\mathrm{wt}_H(f_0(x)(x^4-\Omega_0)^\jmath), \mathrm{wt}_H(f_u(x)(x^4-\Omega_0)^\jmath)\right\} \\ &\geq \mathrm{wt}_H(f_0(x)(x^4-\Omega_0)^\jmath) \\ &\geq d_H(\langle (x^4-\Omega_0)^\jmath\rangle_F), \\ &= d_H(\langle (x^4-\Omega_0)^\chi\rangle_F). \end{aligned}$$

Case 2: When $\Im(x) \neq 0$, then we have

$$\begin{aligned}
&\mathrm{wt}_H(c(x))\\
&\geq \mathrm{wt}_H(u \cdot c(x))\\
&= \mathrm{wt}_H(uf_0(x)(x^4 - \Omega_0)^{\jmath})\\
&\geq d_H(\langle (x^4 - \Omega_0)^{\jmath} \rangle_F),\\
&\geq d_H(\langle (x^4 - \Omega_0)^{\chi} \rangle_F),
\end{aligned}$$

since $\langle (x^4 - \Omega_0)^{\jmath} \rangle_F \subseteq \langle (x^4 - \Omega_0)^{\chi} \rangle_F$. Hence, by combining both the cases, we get $d_H(\langle (x^4 - \Omega_0)^{\chi} \rangle_F) \leq d_H(C)$, which implies that, $d_H(\langle (x^4 - \Omega_0)^{\chi} \rangle_F) = d_H(C)$. $\square$

Next, we compute the Hamming distance of Type 4 Ω-constacyclic codes as follows:

Theorem 5.2.7. *Let* $C = \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x), u(x^4 - \Omega_0)^{\zeta} \rangle$ *be a* Ω*-constacyclic code of Type* 4 *of length* $4p^s$ *over* $\Re$*, where* $\Im(x)$ *is the same as given in Type* 3*,* $\deg(\Im) \leq \zeta - \ell - 1$*,* $\zeta < \chi$*, and* χ *is the smallest possible integer such that* $u(x^4 - \Omega_0)^{\chi} \in \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x) \rangle$*; i.e.,* $\chi = \jmath$*, if* $\Im(x) = 0$ *and otherwise* $\chi = \min\{\jmath, p^s - \jmath + \ell\}$*. Then, we have* $d_H(C) = d_H(\langle (x^4 - \Omega_0)^{\zeta} \rangle_F)$*, and is given by*

$$d_H(C) = (\wp + 1)p^{\sigma},$$

where $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \zeta \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$*,* $1 \leq \wp \leq p - 1$ *and* $0 \leq \sigma \leq s - 1$*.*

Proof. Clearly, we have $C = \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x), u(x^4 - \Omega_0)^{\zeta} \rangle \supseteq \langle u(x^4 - \Omega_0)^{\zeta} \rangle \supseteq \langle u(x^4 - \Omega_0)^{\jmath} \rangle$, since $\zeta < \chi \leq \jmath$. Thus, $d_H(C) \leq d_H(\langle u(x - \Omega_0)^{\zeta} \rangle) = d_H(\langle (x^4 - \Omega_0)^{\zeta} \rangle_F)$. To prove that $d_H(\langle (x^4 - \Omega_0)^{\zeta} \rangle_F) \leq d_H(C)$, we take an arbitrary polynomial $c(x) \in C$ and proceed to show that $\mathrm{wt}_H(c(x)) \geq d_H(\langle (x^4 - \Omega_0)^{\zeta} \rangle_F)$. Now, there exist polynomials $f_0(x), f_u(x), g_0(x)$ and $g_u(x)$ over $\mathbb{F}_{p^m}$ such that

$$\begin{aligned}c(x) &= [f_0(x)+uf_u(x)][(x^4-\Omega_0)^\jmath + u(x^4-\Omega_0)^\ell \Im(x)] \\ &\quad + u(x^4-\Omega_0)^\zeta[g_0(x)+ug_u(x)] \\ &= f_0(x)(x^4-\Omega_0)^\jmath + u[f_0(x)(x^4-\Omega_0)^\ell \Im(x) \\ &\quad + f_u(x)(x^4-\Omega_0)^\jmath + g_0(x)(x^4-\Omega_0)^\zeta] \\ &= f_0'(x)(x^4-\Omega_0)^\zeta + u[f_0(x)(x^4-\Omega_0)^\ell \Im(x) \\ &\quad + g_0'(x)(x^4-\Omega_0)^\zeta]\end{aligned}$$

where $f_0'(x) = f_0(x)(x^4-\Omega_0)^{\jmath-\zeta} \in \mathbb{F}_{p^m}[x], g_0'(x) = f_u(x)(x^4-\Omega_0)^{\jmath-\zeta} + g_0(x) \in \mathbb{F}_{p^m}[x]$. Hence,

$$\begin{aligned}\mathrm{wt}_H(c(x)) &\geq \mathrm{wt}_H(u\cdot c(x)) \\ &\geq \mathrm{wt}_H(uf_0'(x)(x^4-\Omega_0)^\zeta) \\ &\geq d_H(\langle (x^4-\Omega_0)^\zeta\rangle_F)\end{aligned}$$

□

We provide some examples of Λ-constacyclic codes with new parameters of length $4p^s$ over $\Re = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $\Lambda \in \Re^*$.

Example 5.2.8. We present some examples of $(\Phi+u\Psi)$-constacyclic and Ω-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $\Phi, \Psi, \Omega \in \mathbb{F}_p^*$. In Table 6, we consider $p = 5, m = 3, s = 1$ and 2 and in Table 7, we compute examples by considering $p = 13, m = 1, s = 1$ and 2.

Table 6. Λ-constacyclic codes over $\mathbb{F}_{5^3} + u\mathbb{F}_{5^3}$ of length $4p^s$

n	Λ	$\langle g(x)\rangle$	$[n, M, d_H]$
20	$(2+4u)$	$\langle (x^4-2)^2\rangle$	$[20, 5^{96}, 1]$
20	$(3+u)$	$\langle (x^4-3)^8\rangle$	$[20, 5^{24}, 5]$
20	3	$\langle u(x^4-3)\rangle$	$[20, 5^{48}, 2]$
20	2	$\langle (x^4-2)^2+u\rangle$	$[20, 5^{72}, 3]$
20	2	$\langle (x^4-2)^4, u(x^4-2)^3\rangle$	$[20, 5^{36}, 4]$
100	$(3+2u)$	$\langle (x^4-3)^{49}\rangle$	$[100, 5^{12}, 25]$
100	2	$\langle u(x^4-2)^{23}\rangle$	$[100, 5^{24}, 20]$
100	3	$\langle (x^4-3)^3\rangle$	$[100, 5^{528}, 4]$
100	3	$\langle (x^4-3)^{24}+u(x^4-3), u(x^4-3)^{21}\rangle$	$[100, 5^{60}, 10]$

Table 7. **Λ-constacyclic codes over $\mathbb{F}_{13}+u\mathbb{F}_{13}$ of length $4p^s$**

n	Λ	$\langle g(x)\rangle$	$[n, M, d_H]$
52	$(11+7u)$	$\langle (x^4-11)^{25}\rangle$	$[52, 13^4, 13]$
52	$(5+12u)$	$\langle (x^4-5)^2\rangle$	$[52, 13^{96}, 1]$
52	8	$\langle u(x^4-8)^9\rangle$	$[52, 13^{16}, 10]$
52	2	$\langle (x^4-2)^6\rangle$	$[52, 13^{56}, 7]$
52	7	$\langle (x^4-7)^4+u, u(x^4-7)^3\rangle$	$[52, 13^{76}, 4]$
676	$(6+8u)$	$\langle 1\rangle$	$[676, 13^{1352}, 1]$
676	2	$\langle u(x^4-2)^{168}\rangle$	$[676, 13^4, 156]$
676	8	$\langle (x^4-8)^{15}+u(x^4-8)^2\rangle$	$[676, 13^{1232}, 16]$
676	11	$\langle (x^4-11)^{24}, u(x^4-11)^{10}\rangle$	$[676, 13^{5272}, 11]$

5.3 MDS Codes with respect to Hamming Distance

In [5], Norton et al. discussed the Singleton bound for finite chain ring $\Re$ with respect to the Hamming distance $d_H(C)$, which is given as $|C| \leq |\Re|^{n-d_H(C)+1}$.

Corollary 5.3.1. *(Singleton Bound with respect to Hamming Distance)[5] Let C be a linear code of length n over $\Re$ with Hamming distance $d_H(C)$. Then, the Singleton bound with respect to the Hamming distance $d_H(C)$ is given by $|C| \leq p^{2m(n-d_H(C)+1)}$.*

Definition 5.3.2. Let C be a linear code of length n over $\Re$. Then, C is said to be a maximum distance separable (MDS) code with respect to the Hamming distance if it attains the Singleton bound.

In this section, we explore all MDS codes of length $4p^s$. First, we consider the case when Λ is a unit of the form $\Lambda = (\Phi+u\Psi)$. We identify the MDS $(\Phi+u\Psi)$-constacyclic codes of length $4p^s$ as follows:

Theorem 5.3.3. *Let* $C = \langle (x^4 - \Phi_0)^\jmath \rangle \subseteq \frac{\Re[x]}{\langle x^{4p^s} - (\Phi + u\Psi) \rangle}$ *be a* $(\Phi + u\Psi)$*-constacyclic code of length* $4p^s$ *over* $\Re$*. Then, the only MDS code for the Hamming distance is the ambient ring* $\Re_{\Phi+u\Psi} = \frac{\Re[x]}{\langle x^{4p^s} - (\Phi+u\Psi) \rangle}$ *itself.*

Proof. Let C be a $(\Phi + u\Psi)$-constacyclic code of length $4p^s$ over $\Re$. From Theorem 5.2.1, we have $|C| = p^{4m(2p^s - \jmath)}$.

Case 1: When $0 \leq \jmath \leq p^s$, then the Hamming distance is $d_H(C) = 1$ by Theorem 5.2.4. Then, C is MDS if and only if $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{4m(2p^s-\jmath)} = p^{2m(4p^s-1+1)}$ i.e., $4p^s - 2\jmath = 4p^s$ i.e., $\jmath = 0$. Hence, $C = \langle 1 \rangle$ is an MDS $(\Phi + u\Psi)$-constacyclic code of length $4p^s$ over $\Re$.

Case 2: When $2p^s - p^{s-\sigma} + (\wp-1)p^{s-\sigma-1} + 1 \leq \jmath \leq 2p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then the Hamming distance is $d_H(C) = (\wp+1)p^\sigma$. So, C is MDS if and only if $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{4m(2p^s-\jmath)} = p^{2m(4p^s - d_H(C)+1)}$ i.e., $4p^s - 2\jmath = 4p^s - d_H(C) + 1$ i.e., $2\jmath = d_H(C) - 1$. Now, we have

$$\begin{aligned}
\jmath &\geq 2p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \\
&\geq 2p^{\sigma+1} - p + (\wp - 1) + 1 \text{ (equality when } \sigma = s-1) \\
&\geq 2(\wp + 1)p^\sigma - (\wp + 1) + (\wp - 1) + 1 \\
&\quad \text{(equality when } p - 1 = \wp) \\
&= 2d_H(C) - 1. \\
\text{So, } 2\jmath &\geq 2(2d_H(C) - 1) > d_H(C) - 1.
\end{aligned}$$

Thus, MDS code does not exist in this case.

Case 3: When $\jmath = 2p^s$, then $C = \langle 0 \rangle$, and $d_H(C) = 0$. For C to be MDS we must have $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $1 = p^{2m(4p^s+1)}$ i.e., $4p^s + 1 = 0$, which is a not true for any p and s. Thus, there is no MDS code in this case. $\square$

Next, we consider the case when Λ is a unit of the form $\Lambda = \Omega \in \mathbb{F}_{p^m}^*$. In this case, we identify the MDS codes for each type of Ω-constacyclic codes one by one. First, we consider the Ω-constacyclic codes of length $4p^s$ of Type 1.

Theorem 5.3.4. *Let C be a Ω-constacyclic code of Type 1 of length $4p^s$ over $\Re$, then the only MDS code is $\langle 1 \rangle$.*

Proof. **Case 1:** If $C = \langle 0 \rangle$, then the Hamming distance is $d_H(C) = 0$. For C to be MDS we must have, $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $1 = p^{2m(4p^s+1)}$ i.e., $4p^s + 1 = 0$, which is not true for any p and s.

Case 2: If $C = \langle 1 \rangle$, then $d_H(C) = 1$. For C to be MDS we must have $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{8mp^s} = p^{2m(4p^s-1+1)}$, which is true for all p and s. Thus, the code C is MDS in this case. $\square$

Now we examine the MDS condition for Type 2 Ω-constacyclic codes.

Theorem 5.3.5. *Let $C = \langle u(x^4 - \Omega_0)^{\jmath} \rangle$ be a Ω-constacyclic code of Type 2 of length $4p^s$ over $\Re$, where $0 \leq \jmath \leq p^s - 1$, then no MDS codes exist.*

Proof. **Case 1:** If $\jmath = 0$, then $d_H(C) = 1$. For C to be MDS we must have $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{4mp^s} = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{4mp^s} = p^{8mp^s}$, which is not true for any p, m, and s. Thus, C is not MDS for $\jmath = 0$.

Case 2: If $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \jmath \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then we have Hamming distance $d_H = (\wp + 1)p^{\sigma}$. So, C is an MDS code if and only if $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{4m(p^s-\jmath)} = p^{2m(4p^s - d_H(C)+1)}$ i.e., $2\jmath = d_H(C) - 2p^s - 1$.

Now,

$$\begin{aligned}
\jmath &\geq p^{s} - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1\\
&\geq p^{\sigma+1} - p + (\wp - 1) + 1 \text{ (equality when } \sigma = s-1)\\
&\geq (\wp + 1)p^{\sigma} - (\wp + 1) + (\wp - 1) + 1 \text{ (equality when } p - 1 = \wp)\\
&= d_H(C) - 1.
\end{aligned}$$

So, we have

$$\begin{aligned}
2\jmath &\geq 2(d_H(C) - 1)\\
&= 2(d_H(C) - p^{s} - 1) + 2p^{s}\\
&= (d_H(C) - 2p^{s} - 1) + d_H(C) + 2p^{s} - 1\\
&> d_H(C) - 2p^{s} - 1.
\end{aligned}$$

Thus, there is no MDS code in this case. □

Here, we consider the Ω-constacyclic codes of Type 3 to verify the MDS condition for these codes.

Theorem 5.3.6. *Let $C = \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x)\rangle$, be a Ω-constacyclic code of length $4p^s$ over $\Re$, where $1 \leq \jmath \leq p^s - 1$, $0 \leq \ell < \jmath$, and either $\Im(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Omega\rangle}$ or 0. Then, there is no MDS code.*

Proof. We consider two cases as follows:

Case 1: If $\Im(x) = 0$ and $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then we have $d_H(C) = (\wp + 1)p^{\sigma}$. So, C is MDS if and only if $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{8m(p^s - \jmath)} = p^{2m(4p^s - d_H(C)+1)}$ i.e., $4\jmath = d_H(C) - 1$ i.e., $4\chi = d_H(C) - 1$.

Now,

$$\begin{aligned}\chi &\geq p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1\\ &\geq p^{\sigma+1} - p + (\wp - 1) + 1 \text{ (equality when } \sigma = s-1)\\ &\geq (\wp+1)p^{\sigma} - (\wp+1) + (\wp-1) + 1\\ &\quad \text{(equality when } p - 1 = \wp)\\ &= d_H(C) - 1.\end{aligned}$$

So, $4\chi \geq 4(d_H(C) - 1) > d_H(C) - 1$. Thus, MDS code does not exist in this case.

Case 2: If $\Im(x) \neq 0$ and $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, we have the following subcases:

Subcase 2.1: If $1 \leq \jmath \leq \frac{p^s+\ell}{2}$, then $\chi = \jmath$. Also, C is MDS if and only if $|C| = p^{2m(4p^s - d_H(C)+1)}$ i.e., $p^{8m(p^s-\jmath)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $4\jmath = d_H(C) - 1$ i.e., $4\chi = d_H(C) - 1$. Now,

$$\begin{aligned}\chi &\geq p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1\\ &\geq p^{\sigma+1} - p + (\wp - 1) + 1 \text{ (equality when } \sigma = s-1)\\ &\geq (\wp+1)p^{\sigma} - (\wp+1) + (\wp-1) + 1\\ &\quad \text{(equality when } p - 1 = \wp)\\ &= d_H(C) - 1.\end{aligned}$$

So, $\chi \geq 4(d_H(C) - 1) > d_H(C) - 1$. Thus, MDS code does not exist in this case.

Subcase 2.2: If $\frac{p^s+\ell}{2} < \jmath \leq p^s - 1$, then $\chi = p^s - \jmath + \ell$. Also, C is MDS if and only if $|C| = p^{2m(4p^s-d_H(C)+1)}$ i.e., $p^{4m(p^s-\ell)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $2\ell = d_H(C) - 2p^s - 1$ i.e.,

$2p^s + 2\ell = d_H(C) - 1$ i.e., $2p^s - 2\jmath + 2\ell = d_H(C) - 2\jmath - 1$ i.e., $2\chi = d_H(C) - 2\jmath - 1$. Also, we have

$$\begin{aligned} 2\chi &\geq 2(d_H(C) - 1) \\ &\geq 2(d_H(C) - 2\jmath - 1) + 4\jmath \\ &> d_H(C) - 2\jmath - 1. \end{aligned}$$

Thus, there is no MDS code in this case. □

Finally, we explore the MDS Ω-constacyclic codes of Type 4.

Theorem 5.3.7. *Let* $C = \langle (x^4 - \Omega_0)^\jmath + u(x^4 - \Omega_0)^\ell \Im(x), u(x^4 - \Omega_0)^\zeta \rangle$, *be a* Ω-*constacyclic code of Type* 4 *of length* $4p^s$ *over* $\Re$, *where* $1 \leq \jmath \leq p^s - 1$, $0 \leq \ell < \jmath$, *either* $\Im(x)$ *is a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Omega \rangle}$ *or* 0, $\deg(\Im) \leq \zeta - \ell - 1$, $\zeta < \chi$, *and* χ *is the smallest possible integer such that* $u(x^4 - \Omega_0)^\chi \in \langle (x^4 - \Omega_0)^\jmath + u(x^4 - \Omega_0)^\ell \Im(x) \rangle$; *i.e.* $\chi = \jmath$, *if* $\Im(x) = 0$, *otherwise* $\chi = \min\{\jmath, p^s - \jmath + \ell\}$. *Then, there is no MDS code.*

Proof. If $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \zeta \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then Hamming distance is $d_H = (\wp + 1)p^\sigma$. So, C is MDS if and only if $|C| = p^{2m(4p^s - d_H(C) + 1)}$ i.e.,

$p^{4m(2p^s-\jmath-\zeta)} = p^{2m(4p^s-d_H(C)+1)}$ i.e., $2\zeta = d_H(C) - 2\jmath - 1$. Now,

$$\begin{aligned}
\zeta &\geq p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1\\
&\geq p^{\sigma+1} - p + (\wp - 1) + 1 \text{ (equality when } \sigma = s-1)\\
&\geq (\wp + 1)p^{\sigma} - (\wp + 1) + (\wp - 1) + 1\\
&\quad \text{(equality when } p - 1 = \wp)\\
&= d_H(C) - 1.\\
So,\ 2\zeta &\geq 2(d_H(C) - 2\jmath - 1) + 4\jmath\\
&> 2(d_H(C) - 2\jmath - 1)\ (as\ \jmath > 0)\\
&> (d_H(C) - 2\jmath - 1).
\end{aligned}$$

Thus, MDS code does not exist in this case. □

5.4 b-symbol Distance of C over $\Re$

Recently, in [2], Dinh et al. provided the b-symbol distance for $C = \langle (x^{\eta} - \Lambda_0)^{\jmath} \rangle$, for $0 \leq \jmath \leq p^s$, and $b \leq \eta$ over $\mathbb{F}_{p^m}$, where $(x^{\eta} - \Lambda_0)$ is irreducible. We are writing those results for $\eta = 4$ as follows:

Theorem 5.4.1. *[2] Let* $C = \langle (x^4 - \Phi_0)^{\jmath} \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Phi \rangle}$, *for* $0 \leq \jmath \leq p^s$, *then for* $b \leq 4$, *the b-symbol distance distribution $d_b(C)$ is completely given by*

$$d_b(C) = \begin{cases} b, & \text{if } \jmath = 0 \\ b(\wp+1)p^{\sigma}, & \text{if } p^s - p^{s-\sigma} + (\wp-1)p^{s-\sigma-1} \\ & +1 \leq \jmath \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1} \\ 0, & \text{if } \jmath = p^s \end{cases}$$

where $1 \leq \wp \leq p-1$, $0 \leq \sigma \leq s-1$.

The statements and proofs of Theorems 5.2.4, 5.2.5, 5.2.6, 5.2.7 for Hamming distances work the same way for b-symbol distances, providing the b-symbol distances of all $(\Phi + u\Psi)$-constacyclic codes and Ω-constacyclic codes of length $4p^s$ over $\Re$.

Theorem 5.4.2. *Let $C \subseteq \Re_{\Phi+u\Psi} = \frac{\Re[x]}{\langle x^{4p^s} - (\Phi+u\Psi)\rangle}$, then $C = \langle (x^4 - \Phi_0)^{\jmath}\rangle$ for $\jmath \in \{0, 1, \ldots, 2p^s\}$. Then the b-symbol distance distribution $d_b(C)$ is given by*

$$d_b(C) = \begin{cases} b, & \text{if } 0 \leq \jmath \leq p^s \\ b(\wp+1)p^{\sigma}, & \text{if } 2p^s - p^{s-\sigma} + (\wp-1)p^{s-\sigma-1} \\ & +1 \leq \jmath \leq 2p^s - p^{s-\sigma} + \wp p^{s-\sigma-1} \\ 0, & \text{if } \jmath = 2p^s \end{cases}$$

where $1 \leq \wp \leq p-1$, $0 \leq \sigma \leq s-1$.

Now, we determine the b-symbol distance for each type of Ω-constacyclic code of length $4p^s$ in the following theorems.

Obviously, the trivial ideals $\langle 0\rangle$, $\langle 1\rangle$ reside in Type 1, and hence, their b-symbol distances are given by 0 and b respectively.

The b-symbol distance of Type 2 Ω-constacyclic code can be determined as follows:

Theorem 5.4.3. *Let* $C = \langle u(x^4 - \Omega_0)^{\jmath}\rangle$, $0 \le \jmath \le p^s - 1$ *be a* Ω*-constacyclic code of Type* 2 *of length* $4p^s$ *over* $\Re$*. Then, we have* $d_b(C) = d_b(\langle (x^4 - \Omega_0)^{\jmath}\rangle_F)$*, and* $d_b(C)$ *is given by*

$$d_b(C) = \begin{cases} b, & \text{if } \jmath = 0 \\ b(\wp + 1)p^{\sigma}, & \text{if } p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} \\ & +1 \le \jmath \le p^s - p^{s-\sigma} + \wp p^{s-\sigma-1} \end{cases}$$

where $1 \le \wp \le p - 1$, $0 \le \sigma \le s - 1$.

Now, we discuss the b-symbol distance of Type 3 Ω-constacyclic codes of length $4p^s$.

Theorem 5.4.4. *Let* $C = \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x)\rangle$ *be a* Ω*-constacyclic code of Type* 3 *of length* $4p^s$ *over* $\Re$*, where* $1 \le \jmath \le p^s - 1$, $0 \le \ell < \jmath$ *and either* $\Im(x)$ *is a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Omega\rangle}$ *or* 0*. Then, we have* $d_b(C) = d_b(\langle (x^4 - \Omega_0)^{\chi}\rangle_F)$*, where* χ *is the smallest integer such that* $u(x^4 - \Omega_0)^{\chi} \in \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x)\rangle$*, which is given by*

$$\chi = \begin{cases} \jmath, & \text{if } \Im(x) = 0 \\ \min\{\jmath, p^s - \jmath + \ell\}, & \text{if } \Im(x) \neq 0 \end{cases}$$

and is determined by

$$d_b(C) = b(\wp + 1)p^{\sigma},$$

where $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \le \chi \le p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, $1 \le \wp \le p - 1$ *and* $0 \le \sigma \le s - 1$.

Next, we compute the b-symbol distance of Type 4 Ω-constacyclic codes as follows:

Theorem 5.4.5. *Let* $C = \langle (x^4-\Omega_0)^{\jmath} + u(x^4-\Omega_0)^{\ell}\Im(x), u(x^4-\Omega_0)^{\zeta}\rangle$ *be a* Ω*-constacyclic code of Type* 4 *of length* $4p^s$ *over* $\Re$*, where* $\Im(x)$ *is same as given in Type* 3*,* $\deg(\Im) \leq \zeta - \ell - 1$*,* $\zeta < \chi$*, and* χ *is the smallest possible integer such that* $u(x^4-\Omega_0)^{\chi} \in \langle (x^4-\Omega_0)^{\jmath} + u(x^4-\Omega_0)^{\ell}\Im(x)\rangle$*; i.e.,* $\chi = \jmath$*, if* $\Im(x) = 0$ *and otherwise* $\chi = \min\{\jmath, p^s - \jmath + t\}$*. Then, we have* $d_b(C) = d_b(\langle (x^4-\Omega_0)^{\zeta}\rangle_F)$*, and is given by*

$$d_b(C) = b(\wp + 1)p^{\sigma},$$

where $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \zeta \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$*,* $1 \leq \wp \leq p-1$ *and* $0 \leq \sigma \leq s-1$*.*

Example 5.4.6. We present some examples of b-symbol $(\Phi + u\Psi)$-constacyclic and b-symbol Ω-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $\Phi, \Psi, \Omega \in \mathbb{F}_p^*$. In Table 8, we compute the b-symbol distances for $p = 5, m = 2, s = 1$ and $2,$ and $b \leq 4$ and in Table 9, b-symbol distances have been computed by taking $p = 13, m = 1, s = 1$ and $2,$ and $b \leq 4$. We have MDS b-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ denoted by $(*)$ in Table 8 and 9.

Table 8. b**-symbol** Λ**-constacyclic codes over** $\mathbb{F}_{5^2} + u\mathbb{F}_{5^2}$ **of length** $4p^s$

n	b	Λ	$\langle g(x)\rangle$	$[n, M, d_b]$
20	2	$(2+4u)$	$\langle (x^4-2)^2\rangle$	$[20, 5^{64}, 2]$
20	4	$(3+u)$	$\langle (x^4-3)^8\rangle$	$[20, 5^{16}, 20]$
20	3	3	$\langle u(x^4-3)\rangle$	$[20, 5^{32}, 6]$
20	4	2	$\langle (x^4-2)^2 + u\rangle$	$[20, 5^{48}, 12]^*$
20	3	2	$\langle (x^4-2)^4, u(x^4-2)^3\rangle$	$[20, 5^{24}, 12]$
100	3	$(3+2u)$	$\langle (x^4-3)^{49}\rangle$	$[100, 5^8, 75]$
100	2	2	$\langle u(x^4-2)^{23}\rangle$	$[100, 5^{16}, 40]$
100	4	3	$\langle (x^4-3)^{24}\rangle$	$[100, 5^{16}, 100]^*$
100	4	3	$\langle (x^4-3)^{24} + u(x^4-3), u(x^4-3)^{21}\rangle$	$[100, 5^{40}, 40]$

Table 9. b-symbol Λ-constacyclic codes over $\mathbb{F}_{13}+u\mathbb{F}_{13}$ of length $4p^s$

n	b	Λ	$\langle g(x)\rangle$	$[n, M, d_b]$
52	2	$(11+7u)$	$\langle (x^4-11)^{25}\rangle$	$[52, 13^4, 26]$
52	4	$(5+12u)$	$\langle (x^4-5)^2\rangle$	$[52, 13^{96}, 4]$
52	3	8	$\langle u(x^4-8)^9\rangle$	$[52, 13^{16}, 30]$
52	4	2	$\langle (x^4-2)^6\rangle$	$[52, 13^{56}, 28]^*$
52	2	7	$\langle (x^4-7)^4+u, u(x^4-7)^3\rangle$	$[52, 13^{76}, 8]$
676	4	$(6+8u)$	$\langle 1\rangle$	$[676, 13^{1352}, 4]$
676	2	2	$\langle u(x^4-2)^{168}\rangle$	$[676, 13^4, 312]$
676	4	8	$\langle (x^4-8)^1+u\rangle$	$[676, 13^{1344}, 8]^*$
676	2	11	$\langle (x^4-11)^{24}, u(x^4-11)^{10}\rangle$	$[676, 13^{1216}, 22]$

5.5 MDS Codes with respect to b-symbol Distance

Theorem 5.5.1. *(Singleton Bound with respect to b-symbol distance)[6] Let C be a linear b-symbol code of length n over $\Re$ with b-symbol distance $d_b(C)$. Then, the Singleton bound with respect to the b-symbol distance $d_b(C)$ is given by $|C| \leq p^{2m(n-d_b(C)+b)}$.*

Proof. Suppose that C is an $(n, M, d_b(C))$ b-symbol code with $b \leq d_b(C) \leq n$. After deleting the last $d_b(C)-b$ coordinates from all the codewords in C, we observe that any $d_b(C)-b$ consecutive coordinates contribute at most $d_b(C)-1$ to the b-symbol distance. Thus, since C has b-symbol distance $d_b(C)$, so the resulting codewords of length $n-d_b(C)+b$ are still distinct. The conclusion follows from the fact that the maximum number of distinct codewords of length $n-d_b(C)+b$ over $\Re$ is $p^{2m(n-d_b(C)+b)}$. □

Definition 5.5.2. Let C be a b-symbol linear code of length n over $\Re$. Then, C is said to be an MDS b-symbol code with respect to the b-symbol distance if it attains the Singleton bound.

In this section, we explore all MDS b-symbol codes for $b \leq 4$ of length $4p^s$. First, we consider the case when Λ is a unit of the form $\Lambda = (\Phi + u\Psi)$. We identify the MDS b-symbol $(\Phi + u\Psi)$-constacyclic codes of length $4p^s$ as follows:

Theorem 5.5.3. *Let* $C = \langle (x^4 - \Phi_0)^\jmath \rangle \subseteq \frac{\Re[x]}{\langle x^{4p^s} - (\Phi + u\Psi) \rangle}$ *be a* $(\Phi + u\Psi)$*-constacyclic code of length* $4p^s$ *over* $\Re$*. Then, for* $b \leq 4$*, the only MDS* b*-symbol code is the ambient ring* $\Re_{\Phi + u\Psi} = \frac{\Re[x]}{\langle x^{4p^s} - (\Phi + u\Psi) \rangle}$ *itself.*

Proof. Let C be a $(\Phi + u\Psi)$-constacyclic code of length $4p^s$ over $\Re$. From Theorem 5.2.1, we have $|C| = p^{4m(2p^s - \jmath)}$.

Case 1: When $0 \leq \jmath \leq p^s$, then the b-symbol distance is $d_b(C) = b$ by Theorem 5.4.2. Then, C is MDS if and only if $|C| = p^{2m(4p^s - d_b(C) + b)}$ i.e., $p^{4m(2p^s - \jmath)} = p^{2m(4p^s - b + b)}$ i.e., $4p^s - 2\jmath = 4p^s$ i.e., $\jmath = 0$. Hence, $C = \langle 1 \rangle$ is an MDS b-symbol $(\Phi + u\Psi)$-constacyclic code of length $4p^s$ over $\Re$.

Case 2: When $2p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \jmath \leq 2p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then the b-symbol distance is $d_b(C) = b(\wp + 1)p^\sigma$. So, C is MDS if and only if $|C| = p^{2m(4p^s - d_b(C) + b)}$ i.e., $p^{4m(2p^s - \jmath)} = p^{2m(4p^s - d_b(C) + b)}$ i.e., $4p^s - 2\jmath = 4p^s - d_b(C) + b$ i.e.,

$2\jmath = d_b(C) - b$. Now, we have

$$\begin{aligned}
2\jmath &\geq 2(2p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1) \\
&\geq 4p^{\sigma+1} - 2p + 2(\wp - 1) + 2 \text{ (equality when } \sigma = s - 1) \\
&\geq 4(\wp + 1)p^{\sigma} - 2(\wp + 1) + 2(\wp - 1) + 2 \\
&\quad \text{(equality when } p - 1 = \wp) \\
&\geq b(\wp + 1)p^{\sigma} - b + 2 \text{ (equality when } b = 4)
\end{aligned}$$

$$\text{So, } 2\jmath > (d_b(C) - b)$$

Thus, MDS b-symbol code does not exist in this case.

Case 3: When $\jmath = 2p^s$, then $C = \langle 0 \rangle$, and $d_b(C) = 0$. For C to be MDS we must have $|C| = p^{2m(4p^s - d_b(C) + b)}$ i.e., $1 = p^{2m(4p^s+b)}$ i.e., $4p^s + b = 0$, which is not true for any p and s. Thus, there is no MDS b-symbol code in this case. □

Next, we consider the case when Λ is a unit of the form $\Lambda = \Omega \in \mathbb{F}_{p^m}^*$. In this case, we identify the MDS b-symbol codes for each type of Ω-constacyclic codes one by one. First, we consider the b-symbol Ω-constacyclic codes of length $4p^s$ of Type 1.

Theorem 5.5.4. *Let C be a b-symbol Ω-constacyclic code of Type 1 of length $4p^s$ over $\Re$, then, for $b \leq 4$, the only MDS b-symbol code is $\langle 1 \rangle$.*

Proof. **Case 1:** If $C = \langle 0 \rangle$, then the b-symbol distance is $d_b(C) = 0$. For C to be MDS we must have, $|C| = p^{2m(4p^s - d_b(C) + b)}$ i.e., $1 = p^{2m(4p^s+b)}$ i.e., $4p^s + b = 0$, which is not true for any p and s.

Case 2: If $C = \langle 1 \rangle$, then $d_b(C) = b$. For C to be MDS we must have $|C| = p^{2m(4p^s - d_b(C) + b)}$ i.e., $p^{8mp^s} = p^{2m(4p^s - b + b)}$, which is true for all p and s. Thus, the code C is MDS in this case. □

Now we examine the MDS condition for Type 2 b-symbol Ω-constacyclic codes.

Theorem 5.5.5. *Let* $C = \langle u(x^4 - \Omega_0)^{\jmath} \rangle$ *be a b-symbol Ω-constacyclic code of Type* 2 *of length* $4p^s$ *over* $\Re$*, where* $0 \leq \jmath \leq p^s - 1$ *, then, for* $b \leq 4$*, no MDS b-symbol codes exist.*

Proof. **Case 1:** If $\jmath = 0$, then $d_b(C) = b$. For C to be MDS we must have $|C| = p^{2m(4p^s - d_b(C)+b)}$ i.e., $p^{4mp^s} = p^{2m(4p^s - d_b(C)+b)}$ i.e., $p^{4mp^s} = p^{8mp^s}$, which is not true for any p, m, and s. Thus, C is not MDS for $\jmath = 0$.

Case 2: If $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \jmath \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then we have b-symbol distance $d_b(C) = b(\wp + 1)p^{\sigma}$. So, C is an MDS code if and only if $|C| = p^{2m(4p^s - d_b(C)+b)}$ i.e., $p^{4m(p^s - \jmath)} = p^{2m(4p^s - d_b(C)+b)}$ i.e., $2\jmath = d_b(C) - 2p^s - b$.

Now,

$$\begin{aligned}
2\jmath &\geq 2(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1) \\
&\geq 2p^{\sigma+1} - 2p + 2(\wp - 1) + 2 \text{ (equality when } \sigma = s - 1) \\
&\geq 4(\wp + 1)p^{\sigma} - 2p^s - 2(\wp + 1) + 2(\wp - 1) + 2 \\
&\quad \text{(equality when } p - 1 = \wp) \\
&\geq d_b(C) - 2p^s - b + 2 \text{ (equality when } b = 4) \\
&> d_b(C) - 2p^s - b.
\end{aligned}$$

Thus, there is no MDS b-symbol code in this case. □

Here, we consider the b-symbol Ω-constacyclic codes of Type 3 to verify the MDS condition for these codes.

Theorem 5.5.6. *Let* $C = \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell} \Im(x) \rangle$, *be a b-symbol Ω-constacyclic code of length* $4p^s$ *over* $\Re$*, where* $1 \leq \jmath \leq p^s - 1$, $0 \leq \ell < \jmath$, *and either* $\Im(x)$ *is a* *unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s}-\Omega\rangle}$ *or* 0*. Then, for* $b \leq 4$*, there are MDS* b*-symbol code* C *for which* χ *is such that any one of the following conditions holds true:*

- *If* $\Im(x) = 0$
  - *If* $s = 1$*, and* $b = 4$ *then* $d_b(C) = b(\chi + 1)$ *for* $1 \leq \chi \leq p - 1$*.*
  - *If* $s \geq 2$*, then*
    1. $\chi = 1$*, and* $b = 4$ *then* $d_b(C) = 2b$*,*
    2. $\chi = p^s - 1$*, and* $b = 4$ *then* $d_b(C) = bp^s$*.*
- *If* $\Im(x) \neq 0$
  - *If* $s = 1$*, and* $b = 4$ *then* $d_b(C) = b(\chi + 1)$ *for* $1 \leq \chi \leq p - 1$*.*
  - *If* $s \geq 2$*, then*
    1. $\chi = 1$*, and* $b = 4$ *then* $d_b(C) = 2b$*,*
    2. $\chi = p^s - 1$*,* $r = p^s - 2$*, and* $b = 4$ *then* $d_b(C) = bp^s$*.*

.

Proof. We consider two cases as follows:

Case 1: If $\Im(x) = 0$ and $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then we have $d_b(C) = b(\wp + 1)p^{\sigma}$. So, C is MDS if and only if $|C| = p^{2m(4p^s - d_b(C) + b)}$ i.e., $p^{8m(p^s - \jmath)} = p^{2m(4p^s - d_b(C) + b)}$ i.e., $4\jmath = d_b(C) - b$ i.e., $4\chi = d_b(C) - b$.

Now,

$$\begin{aligned}4\chi &\geq 4(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1)\\ &\geq 4(p^{\sigma+1} - p + (\wp - 1) + 1) \text{ (equality when } \sigma = s - 1)\\ &\geq 4(\wp + 1)p^{\sigma} - 4(\wp + 1) + 4(\wp - 1) + 4\\ &\quad \text{(equality when } p - 1 = \wp)\\ &\geq d_b(C) - b \text{ (equality when } b = 4)\end{aligned}$$

So, C is an MDS b-symbol constacyclic code if and only if $s = 1$ and $b = 4$ (in such case, $\jmath = \wp$, $d_b(C) = 4(\wp + 1)$), or $\wp = 1$, $\sigma = 0$ and $b = 4$ (in such case, $\jmath = 1$, $d_b(C) = 2b$), or $\wp = p-1$, $\sigma = s-1$, and $b = 4$ (in such case, $\jmath = p^s - 1$, $d_b(C) = bp^s$).

Case 2: If $\Im(x) \neq 0$ and $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, we have the following subcases:

Subcase 2.1: If $1 \leq \jmath \leq \frac{p^s+\ell}{2}$, then $\chi = \jmath$. Also, C is MDS if and only if $|C| = p^{2m(4p^s - d_b(C)+b)}$ i.e., $p^{8m(p^s - \jmath)} = p^{2m(4p^s - d_b(C)+b)}$ i.e., $4\jmath = d_b(C) - b$ i.e., $4\chi = d_b(C) - b$. Now,

$$\begin{aligned}4\chi &\geq 4(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1)\\ &\geq 4p^{\sigma+1} - 4p + 4(\wp - 1) + 4 \text{ (equality when } \sigma = s - 1)\\ &\geq 4(\wp + 1)p^{\sigma} - 4(\wp + 1) + 4(\wp - 1) + 4\\ &\quad \text{(equality when } p - 1 = \wp)\\ &= d_b(C) - b \text{ (equality when } b = 4)\end{aligned}$$

So, C is an MDS b-symbol constacyclic code if and only if $s = 1$ and $b = 4$ (in such case, $\jmath = \wp$, $d_b(C) = 4(\wp + 1)$), or $\wp = 1$, $\sigma = 0$ and $b = 4$ (in such case, $\jmath = 1$,

$d_b(C) = 2b$), or $\wp = p-1$, $\sigma = s-1$, and $b = 4$ (in such case, $\jmath = p^s - 1$, $\ell = p^s - 2$, $d_b(C) = bp^s$).

Subcase 2.2: If $\frac{p^s+\ell}{2} < \jmath \leq p^s - 1$, then $\chi = p^s - \jmath + \ell$. Also, C is MDS if and only if $|C| = p^{2m(4p^s - d_b(C)+b)}$ i.e., $p^{4m(p^s-\ell)} = p^{2m(4p^s-d_b(C)+b)}$ i.e., $2\ell = d_b(C) - 2p^s - b$ i.e., $2p^s + 2\ell = d_b(C) - b$ i.e., $2p^s - 2\jmath + 2\ell = d_b(C) - 2\jmath - b$ i.e., $2\chi = d_b(C) - 2\jmath - b$. Also, we have

$$\begin{aligned}
2\chi &\geq 2(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1) \\
&\geq 2p^{\sigma+1} - 2p + 2(\wp - 1) + 2 \text{ (equality when } \sigma = s-1) \\
&\geq 4(\wp + 1)p^{\sigma} - 2p^s - 2(\wp + 1) + 2(\wp - 1) + 2 \\
&\quad \text{(equality when } p - 1 = \wp) \\
&\geq d_b(C) - 2p^s - b + 2 \text{ (equality when } b = 4) \\
&\geq d_b(C) - 2(\jmath + 1) - b + 2. \\
&\geq d_b(C) - 2\jmath - b.
\end{aligned}$$

Therefore, C is an MDS b-symbol constacyclic code if and only if $\wp = p-1$, $\sigma = s-1$, and $b = 4$ (in such case, $\jmath = p^s - 1$, $\ell = p^s - 2$, $d_b(C) = bp^s$). $\square$

Finally, we explore the MDS Ω-constacyclic codes of Type 4.

Theorem 5.5.7. *Let* $C = \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x), u(x^4 - \Omega_0)^{\zeta}\rangle$, *be a b-symbol Ω-constacyclic code of Type 4 of length $4p^s$ over $\Re$, where* $1 \leq \jmath \leq p^s - 1$, $0 \leq \ell < \jmath$, *either $\Im(x)$ is a unit in* $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{4p^s} - \Omega\rangle}$ *or* 0, $\deg(\Im) \leq \zeta - \ell - 1$, $\zeta < \chi$, *and χ is the smallest possible integer such that* $u(x^4 - \Omega_0)^{\chi} \in \langle (x^4 - \Omega_0)^{\jmath} + u(x^4 - \Omega_0)^{\ell}\Im(x)\rangle$; *i.e.* $\chi = \jmath$, *if* $\Im(x) = 0$, *otherwise* $\chi = \min\{\jmath, p^s - \jmath + \ell\}$. *Then, there is no MDS b-symbol code for* $b \leq 4$.

Proof. If $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \zeta \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then b-symbol distance is $d_b(C) = b(\wp + 1)p^{\sigma}$. So, C is MDS if and only if $|C| = p^{2m(4p^s - d_b(C)+b)}$ i.e., $p^{4m(2p^s - \jmath - \zeta)} = p^{2m(4p^s - d_b(C)+b)}$ i.e., $2\zeta = d_b(C) - 2\jmath - b$. Now,

$$\begin{aligned}
2\zeta &\geq 2(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1)\\
&\geq 2p^{\sigma+1} - 2p + 2(\wp - 1) + 2 \text{ (equality when } \sigma = s - 1)\\
&\geq 4(\wp + 1)p^{\sigma} - 2p^{\sigma+1} - 2(\wp + 1) + 2(\wp - 1) + 2\\
&\quad \text{(equality when } p - 1 = \wp)\\
&\geq d_b(C) - 2p^s - 2\\
&\quad \text{(equality when } b = 4)\\
&\geq d_b(C) - 2\jmath - b
\end{aligned}$$

So, when $\zeta = p^s - 1, \jmath = p^s - 1$ *and* $b = 4$, then $d_b(C) = d_b(\langle (x^4 - \Omega_0)^{\zeta} \rangle_F) = 4p^s$. Also, we have $d_b(C) - 2\jmath - b = 4p^s - 2(p^s - 1) + 4 = 2(p^s - 1) = 2\zeta$. Thus, equality holds for $\zeta = p^s - 1$.

However, $\zeta < \chi \leq p^s - 1$, which implies that $2\zeta > d_b(C) - 2\jmath - b$.

Therefore, MDS b-symbol code does not exist in this case. □

5.6 Conclusion

The multi-symbol distances of several classes of repeated-root constacyclic codes over the finite ring $\Re$ are determined. Using various types of distance distributions, we explored some novel MDS code classes as an application. For any non-square unit Λ of $\Re$, the Hamming and b-symbol distances of all Λ-constacyclic codes of length $4p^s$ over $\Re$ are completely determined. As examples, several good codes with new parameters are provided. We also identify all MDS constacyclic codes of length $4p^s$ over $\Re$ with respect to the Hamming distance as well as the b-symbol distance. Some non-trivial MDS b-symbol Type 3, Ω-constacyclic codes of length $4p^s$ codes over $\Re$ are constructed with respect to b-symbol distance for $b = 4$.

References

[1] H. Q. Dinh, X. Wang, H. Liu, and S. Sriboonchitta, "On the Hamming distances of repeated-root constacyclic codes of length $4p^s$," *Discrete Mathematics*, vol. 342, no. 5, pp. 1456–1470, 2019.

[2] H. Q. Dinh, X. Wang, and J. Sirisrisakulchai, "Optimal b-symbol constacyclic codes with respect to the Singleton bound," *Journal of Algebra and Its Applications*, vol. 19, no. 08, p. 2 050 151, 2020.

[3] H. Dinh, S Dhompongsa, and S Sriboonchitta, "On constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$," *Discrete Math.*, vol. 340, pp. 832–849, 2017.

[4] S. R. López-Permouth, H. Özadam, F. Özbudak, and S. Szabo, "Polycyclic codes over Galois rings with applications to repeated-root constacyclic codes," *Finite Fields Appl.*, vol. 19, no. 1, pp. 16–38, 2013.

[5] G. H. Norton and A. Sălăgean, "On the structure of linear and cyclic codes over a finite chain ring," *Applicable algebra in engineering, communication and computing*, vol. 10, no. 6, pp. 489–506, 2000.

[6] E. Yaakobi, J. Bruck, and P. H. Siegel, "Constructions and decoding of cyclic codes over b-symbol read channels," *IEEE Trans. Inform. Theory*, vol. 62, no. 4, pp. 1541–1551, 2016.