

# Chapter 4

## On MDS Symbol-Pair Distances of Constacyclic Codes of Length $2p^s$ over $\mathfrak{R}$

### 4.1 Introduction

In this Chapter, we compute the symbol-pair distances of all  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$  for any invertible element  $\Lambda$  of  $\mathfrak{R}$ . We identify all symbol-pair MDS constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$ .

The organization of the present chapter is done as follows. In Section 4.2, we give computational procedures to determine the symbol-pair distances of all  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$ . Also, we give some examples in this section. In next Section 4.3, we explore all symbol-pair MDS codes of length  $2p^s$  over  $\mathfrak{R}$  and give some examples of MDS codes.

## 4.2 Symbol-pair Distances of Codes over $\mathfrak{R}$

In this section, we obtain the symbol-pair distances of all  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$ , where  $\Lambda$  is an arbitrary unit of  $\mathfrak{R}$ . As in [2], we have seen that the structure of  $\Lambda$ -constacyclic codes depends on the types of  $\Lambda$ . So, we have to consider two cases:  $\Lambda$  is a square unit and  $\Lambda$  is a non-square unit.

In [2], Chen et al. obtained the structures of all  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$  as follows.

**Theorem 4.2.1.** [2] *Let  $\Lambda \in \mathfrak{R}$  be an arbitrary unit of  $\mathfrak{R}$ .*

1. *If  $\Lambda$  is a square in  $\mathfrak{R}$ , say  $\Lambda = \Upsilon^2$ , then from the Chinese remainder theorem (CRT),*

$$\mathfrak{R}_\Lambda = \frac{\mathfrak{R}[x]}{\langle x^{2p^s} - \Lambda \rangle} \cong \frac{\mathfrak{R}[x]}{\langle x^{p^s} + \Upsilon \rangle} \oplus \frac{\mathfrak{R}[x]}{\langle x^{p^s} - \Upsilon \rangle}.$$

*This implies that every  $\Lambda$ -constacyclic code of length  $2p^s$  can be represented as a direct sum of an  $-\Upsilon$ -constacyclic code and an  $\Upsilon$ -constacyclic code of length  $p^s$  over  $\mathfrak{R}$ . Note that the algebraic structures of all constacyclic codes of length  $p^s$  over  $\mathfrak{R}$  have been determined in [3].*

2. *Suppose that  $\Lambda$  is a non-square.*

- (a) *If  $\Lambda$  is of form  $(\Phi + u\Psi)$ , then  $\mathfrak{R}_{\Phi+u\Psi} = \frac{\mathfrak{R}[x]}{\langle x^{2p^s} - (\Phi + u\Psi) \rangle}$  is a finite chain ring having maximal ideal  $\langle (x^2 - \Phi_0) \rangle$ , where  $\Phi_0 \in \mathbb{F}_{p^m}$ , satisfying  $\Phi = \Phi_0^{p^s}$ . Also, in  $\mathfrak{R}_{\Phi+u\Psi}$ , we have  $\langle (x^2 - \Phi_0)^{p^s} \rangle = \langle u \rangle$ . Therefore, the structures of  $(\Phi + u\Psi)$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$  are the ideals  $\langle (x^2 - \Phi_0)^j \rangle$ ,  $0 \leq j \leq 2p^s$  of  $\mathfrak{R}_{\Phi+u\Psi}$ . The number of codewords of  $\mathcal{C}$  is  $|\mathcal{C}| = p^{2m(2p^s - j)}$ .*
- (b) *If  $\Lambda$  is of type  $\Omega$ , then  $\mathfrak{R}_\Omega = \frac{\mathfrak{R}[x]}{\langle x^{2p^s} - \Omega \rangle}$  is a local finite non-chain ring with the non-principal maximal ideal  $\langle u, x^2 - \Omega_0 \rangle$ , where  $\Omega_0 \in \mathbb{F}_{p^m}$  satisfying*

$\Omega = \Omega_0^{p^s}$ . Therefore, the structures of  $\Omega$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$  are the ideals of  $\mathfrak{R}_\Omega$ , and are given by:

- Type 1:  $\langle 1 \rangle, \langle 0 \rangle$ , which are trivial ideals. The number of codewords are respectively  $p^{4mp^s}$  and 1.
- Type 2:  $\langle u(x^2 - \Omega_0)^j \rangle$ , where  $0 \leq j \leq p^s - 1$  i.e., Type 2 contains non-monic polynomial generated by principal ideals. The number of codewords of  $\mathcal{C}$  is  $|\mathcal{C}| = p^{2m(p^s-j)}$ .
- Type 3:  $\langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ , where  $1 \leq j \leq p^s - 1$ ,  $0 \leq \ell < j$ , and either  $\mathfrak{S}(x)$  is a unit in  $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{2p^s} - \Omega \rangle}$  or 0 i.e., Type 3 consists monic polynomial generated by principal ideals. Also, the number of codewords is  $|\mathcal{C}| = p^{4m(p^s-j)}$ , when  $\mathfrak{S}(x) = 0$  and

$$|\mathcal{C}| = \begin{cases} p^{4m(p^s-j)}, & \text{if } 1 \leq j \leq p^{s-1} + \frac{\ell}{2} \\ p^{2m(p^s-\ell)}, & \text{if } p^{s-1} + \frac{\ell}{2} + 1 \leq j \leq p^s - 1, \end{cases}$$

when  $\mathfrak{S}(x) \neq 0$ .

- Type 4:  $\langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x), u(x^2 - \Omega_0)^\zeta \rangle$ , where  $\mathfrak{S}(x)$  is the same as in Type 3,  $\deg(\mathfrak{S}) \leq \zeta - \ell - 1$ ,  $\zeta < \chi$ , and  $\chi$  is the smallest possible integer such that  $u(x^2 - \Omega_0)^\chi \in \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ ; i.e.,  $\chi = j$ , if  $\mathfrak{S}(x) = 0$ , otherwise  $\chi = \min\{j, p^s - j + \ell\}$  i.e. non-principal ideals. Also, the number of codewords is  $|\mathcal{C}| = p^{2m(2p^s-j-\zeta)}$ .

In [7], Dinh et al. obtained the symbol-pair distances of all the repeated-root constacyclic codes of length  $p^s$  over  $\mathbb{F}_{p^m}$ . Later, in [6] and [9], the symbol-pair distances of all  $\Lambda$ -constacyclic codes over  $\mathbb{F}_{p^m}$  of length  $2p^s$ , where  $\Lambda$  is a non-square unit, are completely computed.

**Theorem 4.2.2.** [6, 9] Let  $\mathcal{C} = \langle (x^2 - \Phi_0)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{2p^s} - \Phi \rangle}$ , for  $0 \leq j \leq p^s$ , then the symbol-pair distances are completely obtained by

$$d_{sp}(\mathcal{C}) = \begin{cases} 2, & \text{if } j = 0 \\ 2(\wp + 1)p^\sigma, & \text{if } p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} \\ & + 1 \leq j \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1} \\ 0, & \text{if } j = p^s \end{cases}$$

where,  $0 \leq \sigma \leq s - 1$ ,  $1 \leq \wp \leq p - 1$ .

Dinh et al. determined the symbol-pair distances of all repeated-root constacyclic codes of length  $p^s$  over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ , in [4, 8]. The following results are rectified form of those results. From now, we will denote  $d_{sp}(C_F)$  as the symbol-pair distance of the code  $C$  over  $\mathbb{F}_{p^m}$ .

**Theorem 4.2.3.** [4, 8] Let  $\mathcal{C}$  be a  $(\Phi + u\Psi)$ -constacyclic code of length  $p^s$  over  $\mathfrak{R}$ , then  $\mathcal{C} = \langle (x - \Phi_0)^j \rangle \subseteq \frac{\mathfrak{R}[x]}{\langle x^{p^s} - (\Phi + u\Psi) \rangle}$ , for  $j \in \{0, 1, \dots, 2p^s\}$ , and the symbol-pair

distance  $d_{sp}(C)$  is completely determined by

$$d_{sp}(C) = \begin{cases} 2, & \text{if } 0 \leq j \leq p^s \\ 3p^\sigma, & \text{if } j = 2p^s - p^{s-\sigma} + 1, \quad \text{where } 0 \leq \sigma \leq s-2 \\ 4p^\sigma, & \text{if } 2p^s - p^{s-\sigma} + 2 \leq j \leq 2p^s - p^{s-\sigma} + p^{s-\sigma+1} \\ 2(\wp + 1)p^\sigma, & \\ & \text{if } 2p^s - pr + (\wp - 1)r + 1 \leq j \leq 2p^s - pr + \wp r, \\ & \text{where } r = p^{s-\sigma-1}, 0 \leq \sigma \leq s-2 \text{ and } 2 \leq \wp \leq p-1 \\ (\wp + 1)p^{s-1}, & \text{if } j = 2p^s - p + \wp - 1, \text{ where } 1 \leq \wp \leq p-1 \\ p^s, & \text{if } j = 2p^s - 1. \\ 0, & \text{if } j = 2p^s. \end{cases}$$

**Theorem 4.2.4.** [4, 8] Let  $\Lambda = \Omega \in \mathbb{F}_{p^m}^*$  be a unit in  $\mathfrak{R}$ . The  $\Omega$ -constacyclic codes of length  $p^s$  over  $\mathfrak{R}$ , i.e., ideals of the ring  $\frac{\mathfrak{R}[x]}{\langle x^{p^s} - \Omega \rangle}$ , have their symbol-pair distances completely determined as follows.

- Type 1:  $\langle 1 \rangle, \langle 0 \rangle, ; d_{sp}(\langle 1 \rangle) = 2, d_{sp}(\langle 0 \rangle) = 0.$
- Type 2:  $\mathcal{C} = \langle u(x - \Omega_0)^j \rangle$ , where  $0 \leq j \leq p^s - 1;$   
 $d_{sp}(\mathcal{C}) = d_{sp}(\langle (x - \Omega_0)^j \rangle_F)$

$$= \begin{cases} 2, & \text{if } j = 0 \\ 3p^\sigma, & \text{if } j = p^s - p^{s-\sigma} + 1, \quad \text{where } 0 \leq \sigma \leq s - 2 \\ 4p^\sigma, & \text{if } p^s - p^{s-\sigma} + 2 \leq j \leq p^s - p^{s-\sigma} + p^{s-\sigma+1}, \\ 2(\wp + 1)p^\sigma, & \text{if } p^s - pr + (\wp - 1)r + 1 \leq j \leq p^s - pr + \wp r, \\ & \text{where } r = p^{s-\sigma-1}, 0 \leq \sigma \leq s - 2 \text{ and } 2 \leq \wp \leq p - 1 \\ (\wp + 1)p^{s-1}, & \text{if } j = p^s - p + \wp - 1, \text{ where } 1 \leq \wp \leq p - 1 \\ p^s, & \text{if } j = p^s - 1. \end{cases}$$

- *Type 3:  $\mathcal{C} = \langle (x - \Omega_0)^j + u(x - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ , where  $1 \leq j \leq p^s - 1$ ,  $0 \leq \ell < j$ , and either  $\mathfrak{S}(x)$  is 0 or  $\mathfrak{S}(x)$  is a unit in  $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{p^s} - \Omega \rangle}$ ;*  
 $d_{sp}(\mathcal{C}) = d_{sp}(\langle (x - \Omega_0)^\chi \rangle_F)$

$$= \begin{cases} 3p^\sigma, & \text{if } \chi = p^s - p^{s-\sigma} + 1, \text{ where } 0 \leq \sigma \leq s - 2 \\ 4p^\sigma, & \text{if } p^s - p^{s-\sigma} + 2 \leq \chi \leq p^s - p^{s-\sigma} + p^{s-\sigma+1}, \\ 2(\wp + 1)p^\sigma, & \text{if } p^s - pr + (\wp - 1)r + 1 \leq \chi \leq p^s - pr + \wp r, \\ & \text{where } r = p^{s-\sigma-1}, 0 \leq \sigma \leq s - 2 \text{ and } 1 \leq \wp \leq p - 2 \\ (\wp + 1)p^{s-1}, & \text{if } \chi = p^s - p + \wp - 1, \text{ where } 1 \leq \wp \leq p - 1 \\ p^s, & \text{if } \chi = p^s - 1. \end{cases}$$

where,  $\chi$  is the smallest integer such that  $u(x - \Omega_0)^\chi \in \langle (x - \Omega_0)^j + u(x - \Omega_0)^\ell \mathfrak{S}(x) \rangle$  and is given by

$$\chi = \begin{cases} \min\{j, p^s - j + \chi\}, & \text{if } \mathfrak{S}(x) \neq 0 \\ j, & \text{if } \mathfrak{S}(x) = 0 \end{cases}$$

- *Type 4:*  $\langle (x - \Omega_0)^j + u(x - \Omega_0)^\ell \mathfrak{S}(x), u(x - \Omega_0)^\zeta \rangle$ , with  $\mathfrak{S}(x)$  as in Type 3,  $\deg(\mathfrak{S}) \leq \zeta - \ell - 1$ , and  $\zeta < \chi$ , where  $\chi$  is the smallest integer such that  $u(x - \Omega_0)^\chi \in \langle (x - \Omega_0)^j + u(x - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ ; i.e., such  $\chi$  can be determined as

$$\chi = \begin{cases} \min\{j, p^s - j + \ell\}, & \text{if } \mathfrak{S}(x) \neq 0 \\ j, & \text{if } \mathfrak{S}(x) = 0. \end{cases}$$

Then  $d_{sp}(\mathcal{C}) = d_{sp}(\langle (x - \Omega_0)^\zeta \rangle_F)$

$$= \begin{cases} 3p^\sigma, & \text{if } \zeta = p^s - p^{s-\sigma} + 1, \text{ where } 0 \leq \sigma \leq s - 2 \\ 4p^\sigma, & \text{if } p^s - p^{s-\sigma} + 2 \leq \zeta \leq p^s - p^{s-\sigma} + p^{s-\sigma+1}, \\ 2(\wp + 1)p^\sigma, & \text{if } p^s - pr + (\wp - 1)r + 1 \leq \zeta \leq p^s - pr + \wp r, \\ & \text{where } r = p^{s-\sigma-1}, 0 \leq \sigma \leq s - 2 \text{ and } 2 \leq \wp \leq p - 1 \\ (\wp + 1)p^{s-1}, & \text{if } \zeta = p^s - p + \wp - 1, \text{ where } 1 \leq \wp \leq p - 1 \end{cases}$$

### 4.2.1 When $\Lambda$ is a Square Unit of $\mathfrak{R}$

Here, we will compute the symbol-pair distances of all  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$ , where  $\Lambda$  is a square unit in  $\mathfrak{R}$ . We first have the following lemma.

**Lemma 4.2.5.** *Let us consider two non-zero symbol-pair codes  $\mathcal{A} = [n_1, M_1, d_{sp}(\mathcal{A})]$  and  $\mathcal{B} = [n_2, M_2, d_{sp}(\mathcal{B})]$  over  $\mathfrak{R}$ . Then the direct sum of  $\mathcal{A}$  and  $\mathcal{B}$  is defined by*

$$\mathcal{A} \oplus \mathcal{B} = \{(a, b) \mid a \in \mathcal{A} \text{ and } b \in \mathcal{B}\},$$

which is an  $[n_1 + n_2, M_1 M_2, \min\{d_{sp}(\mathcal{A}), d_{sp}(\mathcal{B})\}]$  symbol-pair code over  $\mathfrak{R}$ .

*Proof.* It is clear that  $\mathcal{A} \oplus \mathcal{B}$  is a symbol-pair code of length  $n_1 + n_2$  over  $\mathfrak{R}$ . It is

easy to verify that the size of  $\mathcal{A} \oplus \mathcal{B}$  is equal to the product of the size of  $\mathcal{A}$  and that of  $\mathcal{B}$ , so we have size of  $\mathcal{A} \oplus \mathcal{B}$  is  $M_1 M_2$ .

Now, we may assume that  $d_{sp}(\mathcal{A}) \leq d_{sp}(\mathcal{B})$ . Let  $y = ((y_0, y_1), (y_1, y_2) \dots (y_{n-1}, y_0)) \in \mathcal{A}$  and  $\mathbf{0} = ((0, 0), (0, 0) \dots (0, 0)) \in \mathcal{B}$  with  $wt_{sp}(y) = d_{sp}(\mathcal{A})$ . Then,  $(y, \mathbf{0}) \in \mathcal{A} \oplus \mathcal{B}$ . Hence,  $d_{sp}(\mathcal{A} \oplus \mathcal{B}) \leq wt_{sp}((y, \mathbf{0})) = d_{sp}(\mathcal{A})$ . On other hand, for any nonzero symbol-pair codeword  $(a, b) \in \mathcal{A} \oplus \mathcal{B}$  with  $a = ((a_0, a_1), (a_1, a_2) \dots (a_{n-1}, a_0)) \in \mathcal{A}$  and  $b = ((b_0, b_1), (b_1, b_2), \dots (b_{n-1}, b_0)) \in \mathcal{B}$  with either  $a \neq 0$  or  $b \neq 0$  or both are nonzero codewords. Then,  $wt_{sp}((a, b)) = wt_{sp}(a) + wt_{sp}(b) \geq d_{sp}(\mathcal{A})$ .

In a similar way one can easily show that if  $d_{sp}(\mathcal{B}) \leq d_{sp}(\mathcal{A})$  then  $d_{sp}(\mathcal{A} \oplus \mathcal{B}) = d_{sp}(\mathcal{B})$ . Thus, we get the result.  $\square$

For any arbitrary square invertible element  $\Lambda = \Upsilon^2$  of the ring  $\mathfrak{R}$ , we can determine the symbol-pair distances of the  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$  by the following theorem.

**Theorem 4.2.6.** *Let  $\Lambda = \Upsilon^2 \in \mathfrak{R}$ , and  $\mathcal{C} \cong \mathcal{C}_1 \oplus \mathcal{C}_2$  be a constacyclic code of length  $2p^s$  over  $\mathfrak{R}$ , where  $\mathcal{C}_1, \mathcal{C}_2$  are non-zero ideals of  $\frac{\mathfrak{R}[x]}{\langle x^{p^s} + \Upsilon \rangle}, \frac{\mathfrak{R}[x]}{\langle x^{p^s} - \Upsilon \rangle}$ , respectively. Then, the symbol-pair distance  $d_{sp}(\mathcal{C})$  is  $\min\{d_{sp}(\mathcal{C}_1), d_{sp}(\mathcal{C}_2)\}$ , where,  $d_{sp}(\mathcal{C}_1)$  and  $d_{sp}(\mathcal{C}_2)$  are given by Theorems 4.2.3 and 4.2.4.*

*Proof.* The result immediately follows from Lemma 4.2.5.  $\square$

Next, we consider the symbol-pair distance for non-square units of  $\mathfrak{R}$ .

## 4.2.2 When $\Lambda$ is a Non-square Unit of $\mathfrak{R}$

Throughout this subsection, the unit  $\Lambda$  of  $\mathfrak{R}$  is a non-square.

When  $\Lambda = \Phi + u\Psi$ , then the symbol-pair distance distribution  $d_{sp}(\mathcal{C})$  of  $(\Phi + u\Psi)$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$  is completely determined as follows:

**Theorem 4.2.7.** Let  $\mathcal{C} \subseteq \mathfrak{R}_{\Phi+u\Psi} = \frac{\mathfrak{R}[x]}{\langle x^{2p^s} - (\Phi+u\Psi) \rangle}$ , then  $\mathcal{C} = \langle (x^2 - \Phi_0)^j \rangle$ , for  $j \in \{0, 1, \dots, 2p^s\}$ , and the symbol-pair distances  $d_{sp}(\mathcal{C})$  are given by

$$d_{sp}(\mathcal{C}) = \begin{cases} 2, & \text{if } 0 \leq j \leq p^s \\ 2(\wp + 1)p^\sigma, & \text{if } 2p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} \\ & + 1 \leq j \leq 2p^s - p^{s-\sigma} + \wp p^{s-\sigma-1} \\ 0, & \text{if } j = 2p^s, \end{cases}$$

where,  $0 \leq \sigma \leq s - 1$ ,  $1 \leq \wp \leq p - 1$ .

*Proof.* We consider the following cases:

**Case 1:** When  $j = 2p^s$  and  $j = 0$ , then codes  $\langle 0 \rangle$  and  $\langle 1 \rangle$  have  $d_{sp}(\mathcal{C}) = 0$  and 2 respectively.

**Case 2:** When  $1 \leq j \leq p^s$ . The ring  $\mathfrak{R}_{\Phi+u\Psi}$  is a chain ring whose ideals are  $\mathfrak{R}_{\Phi+u\Psi} = \langle 1 \rangle \supsetneq \langle (x^2 - \Phi_0) \rangle \supsetneq \dots \supsetneq \langle (x^2 - \Phi_0)^{p^s} \rangle \supsetneq \dots \supsetneq \langle (x^2 - \Phi_0)^{2p^s} \rangle = \langle 0 \rangle$ . Clearly,  $u \in \langle (x^2 - \Phi_0)^j \rangle$  and hence we have symbol-pair distance,  $d_{sp}(\mathcal{C}) = 2$ .

**Case 3:** When  $p^s + 1 \leq j \leq 2p^s - 1$ . Then,  $\langle (x^2 - \Phi_0)^j \rangle = \langle u(x^2 - \Phi_0)^{j-p^s} \rangle$ . So, clearly the codewords of the code  $\langle (x^2 - \Phi_0)^j \rangle$  in  $\mathfrak{R}_{\Phi+u\Psi}$  are exactly same as the codewords of the code  $\langle (x^2 - \Phi_0)^{j-p^s} \rangle$  in  $\frac{\mathbb{F}_p^m[x]}{\langle x^{2p^s} - \Phi \rangle}$  multiplied by  $u$ , having the same symbol-pair weights. Moreover, the codes  $\langle (x^2 - \Phi_0)^{j-p^s} \rangle$  of length  $2p^s$  have symbol-pair distances given in Theorem 4.2.2.  $\square$

Now, we compute the symbol-pair distance for each type of  $\Omega$ -constacyclic code of length  $2p^s$  by the following theorems. Clearly,  $\langle 1 \rangle$ ,  $\langle 0 \rangle$  reside in Type 1, and hence, their symbol-pair distances are respectively, 2 and 0. Now, the symbol-pair distance of Type 2  $\Omega$ -constacyclic code can be determined as:

**Theorem 4.2.8.** Let  $\mathcal{C} = \langle u(x^2 - \Omega_0)^j \rangle$ ,  $0 \leq j \leq p^s - 1$ , be a  $\Omega$ -constacyclic code of Type 2 of length  $2p^s$  over  $\mathfrak{R}$ . Then, we have  $d_{sp}(\mathcal{C}) = d_{sp}(\langle (x^2 - \Omega_0)^j \rangle_F)$ , and  $d_{sp}(\mathcal{C})$  is given by

$$d_{sp}(\mathcal{C}) = \begin{cases} 2, & \text{if } j = 0 \\ 2(\wp + 1)p^\sigma, & \text{if } p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} \\ & + 1 \leq j \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}, \end{cases}$$

where,  $0 \leq \sigma \leq s - 1$ ,  $1 \leq \wp \leq p - 1$ .

*Proof.* We have following two cases:

**Case 1:** When  $j = 0$ , then clearly,  $d_{sp}(\mathcal{C}) = 2$ .

**Case 2:** When  $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq j \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ , then for a code  $\mathcal{C} = \langle u(x^2 - \gamma_0)^j \rangle$ ,  $1 \leq j \leq p^s - 1$ , the codewords of  $\mathcal{C}$  are exactly same as the codewords of the  $\Omega$ -constacyclic codes  $\langle (x^2 - \Omega_0)^j \rangle$  in  $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{2p^s} - \Omega \rangle}$  multiplied by  $u$ . Thus, we have,  $d_{sp}(\mathcal{C}) = d_{sp}(\langle (x^2 - \Omega_0)^j \rangle_F)$  and are given by Theorem 4.2.2.  $\square$

Now, we determine the symbol-pair distance of Type 3  $\Omega$ -constacyclic codes of length  $2p^s$ .

**Theorem 4.2.9.** Let  $\mathcal{C} = \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$  be a  $\Omega$ -constacyclic code of Type 3 of length  $2p^s$  over  $\mathfrak{R}$ , where,  $1 \leq j \leq p^s - 1$ ,  $0 \leq \ell < j$  and either  $\mathfrak{S}(x)$  is a unit in  $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{2p^s} - \Omega \rangle}$  or 0. Then, we have  $d_{sp}(\mathcal{C}) = d_{sp}(\langle (x^2 - \Omega_0)^\chi \rangle_F)$ , where  $\chi$  is the smallest integer such that  $u(x^2 - \Omega_0)^\chi \in \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ , and is determined by

$$\chi = \begin{cases} \min\{j, p^s - j + \chi\}, & \text{if } \mathfrak{S}(x) \neq 0 \\ j, & \text{if } \mathfrak{S}(x) = 0 \end{cases}$$

and

$$d_{sp}(\mathcal{C}) = 2(\wp + 1)p^\sigma,$$

where,  $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ ,  $0 \leq \sigma \leq s - 1$  and  $1 \leq \wp \leq p - 1$ .

*Proof.* Since  $\chi$  is the smallest integer such that  $u(x^2 - \Omega_0)^\chi \in \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ , therefore we have,

$$d_{sp}(\mathcal{C}) \leq d_{sp}(\langle u(x^2 - \Omega_0)^\chi \rangle) = d_{sp}(\langle (x^2 - \Omega_0)^\chi \rangle_F).$$

Now, consider an arbitrary polynomial  $c(x) \in \mathcal{C}$  then, there exist two polynomials,  $f_u(x)$  and  $f_0(x)$  over  $(\mathbb{F}_{p^m}, \mathbb{F}_{p^m})$  so that

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x)] \\ &= f_0(x)(x^2 - \Omega_0)^j + u[f_0(x)(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \\ &\quad + f_u(x)(x^2 - \Omega_0)^j]. \end{aligned}$$

We have the following cases:

**Case 1:** When  $\mathfrak{S}(x) = 0$ , then

$$\begin{aligned} &\text{wt}_{sp}(c(x)) \\ &\geq \max\{\text{wt}_{sp}(f_0(x)(x^2 - \Omega_0)^j), \text{wt}_{sp}(f_u(x)(x^2 - \Omega_0)^j)\} \\ &\geq \max\{\text{wt}_{sp}(f_0(x)(x^2 - \Omega_0)^j), \text{wt}_{sp}(f_0(x)(x^2 - \Omega_0)^j)\} \\ &\geq d_{sp}(\langle (x^2 - \Omega_0)^j \rangle_F), \\ &= d_{sp}(\langle (x^2 - \Omega_0)^\chi \rangle_F). \end{aligned}$$

**Case 2:** When  $\mathfrak{S}(x) \neq 0$ , then

$$\begin{aligned}
& \text{wt}_{sp}(c(x)) \\
& \geq \text{wt}_{sp}(u \cdot c(x)) \\
& \geq \text{wt}_{sp}(uf_0(x)(x^2 - \Omega_0)^j) \\
& = d_{sp}(\langle (x^2 - \Omega_0)^j \rangle_F), \\
& = d_{sp}(\langle (x^2 - \Omega_0)^\chi \rangle_F),
\end{aligned}$$

since  $\langle (x^2 - \Omega_0)^j \rangle_F \subseteq \langle (x^2 - \Omega_0)^\chi \rangle_F$ .

Hence, by Cases 1 and 2, we have  $d_{sp}(\langle (x^2 - \Omega_0)^\chi \rangle_F) \leq d_{sp}(\mathcal{C})$ , which gives,  $d_{sp}(\langle (x^2 - \Omega_0)^\chi \rangle_F) = d_{sp}(\mathcal{C})$ .  $\square$

Next, we determine the symbol-pair distance of Type 4  $\Omega$ -constacyclic codes as:

**Theorem 4.2.10.** *Consider  $\mathcal{C} = \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x), u(x^2 - \Omega_0)^\zeta \rangle$ , which is a  $\Omega$ -constacyclic code of Type 4 of length  $2p^s$  over  $\mathfrak{R}$ , where  $\mathfrak{S}(x)$  is the same as given in Type 3,  $\deg(\mathfrak{S}) \leq \zeta - \ell - 1$ ,  $\zeta < \chi$ , and  $\chi$  is the smallest possible integer such that  $u(x^2 - \Omega_0)^\chi \in \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ ; i.e.,  $\chi = \min\{j, p^s - j + \ell\}$ , if  $\mathfrak{S}(x) \neq 0$  and otherwise  $\chi = j$ . Then,  $d_{sp}(\mathcal{C}) = d_{sp}(\langle (x^2 - \Omega_0)^\zeta \rangle_F)$ , and is determined by*

$$d_{sp}(\mathcal{C}) = 2(\wp + 1)p^\sigma,$$

where  $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \zeta \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ ,  $0 \leq \sigma \leq s - 1$ , and  $1 \leq \wp \leq p - 1$ .

*Proof.* Clearly, we have  $\mathcal{C} = \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x), u(x^2 - \Omega_0)^\zeta \rangle \supseteq \langle u(x^2 - \Omega_0)^\zeta \rangle \supseteq \langle u(x^2 - \Omega_0)^j \rangle$ , since  $\zeta < \chi \leq j$ . Thus,  $d_{sp}(\mathcal{C}) \leq d_{sp}(\langle u(x^2 - \Omega_0)^\zeta \rangle) = d_{sp}(\langle (x^2 - \Omega_0)^\zeta \rangle_F)$ . To prove that  $d_{sp}(\langle (x^2 - \Omega_0)^\zeta \rangle_F) \leq d_{sp}(\mathcal{C})$ , we take an arbitrary

polynomial  $c(x) \in \mathcal{C}$  and we will proceed to show that  $\text{wt}_{sp}(c(x)) \geq d_{sp}(\langle (x^2 - \Omega_0)^\zeta \rangle_F)$ .

There exist polynomials  $g_u(x), g_0(x), f_u(x)$  and  $f_0(x)$  over  $\mathbb{F}_{p^m}^2$  such that

$$\begin{aligned}
c(x) &= [f_0(x) + uf_u(x)][(x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x)] \\
&\quad + u(x^2 - \Omega_0)^\zeta [g_0(x) + ug_u(x)] \\
&= f_0(x)(x^2 - \Omega_0)^j + u[f_0(x)(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \\
&\quad + f_u(x)(x^2 - \Omega_0)^j + g_0(x)(x^2 - \Omega_0)^\zeta] \\
&= f'_0(x)(x^2 - \Omega_0)^\zeta + u[f_0(x)(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \\
&\quad + g'_0(x)(x^2 - \Omega_0)^\zeta]
\end{aligned}$$

where,  $g'_0(x) = g_0(x) + f_u(x)(x^2 - \Omega_0)^{j-\zeta}$ ,  $f'_0(x) = f_0(x)(x^2 - \Omega_0)^{j-\zeta} \in \mathbb{F}_{p^m}[x]$ .

Hence,

$$\begin{aligned}
\text{wt}_{sp}(c(x)) &\geq \text{wt}_{sp}(u \cdot c(x)) \\
&\geq \text{wt}_{sp}(uf'_0(x)(x^2 - \Omega_0)^\zeta) \\
&\geq d_{sp}(\langle (x^2 - \Omega_0)^\zeta \rangle_F),
\end{aligned}$$

where,  $h'(x) = g'_0(x)(x^2 - \Omega_0)^\zeta + f_0(x)(x^2 - \Omega_0)^\ell \mathfrak{S}(x)$ . □

**Example 4.2.11.** Here are some examples of symbol-pair  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ . In Table 3, we take square unit  $\Lambda$  and in Table 4, non-square unit  $\Lambda$ . We denote the primitive element of  $\mathbb{F}_{p^m}$  by  $\omega$ .

**Table 3.**  $\Lambda$ -constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ , for square  $\Lambda$ 

$(p, m)$	$n$	$\Lambda$	$\mathcal{C}_1 = \langle g_1(x) \rangle$	$\mathcal{C}_2 = \langle g_2(x) \rangle$	Parameters [ $n, M, d_{sp}$ ] of $\mathcal{C}_1 \oplus \mathcal{C}_2$
(5, 1)	10	$(1 + 4u)$	$\langle (x + 1)^7 \rangle$	$\langle (x - 1)^4 \rangle$	[10, $5^9$ , 2]
(13, 2)	26	$(3 + 11u)$	$\langle (x + 4)^{15} \rangle$	$\langle (x - 4)^{25} \rangle$	[26, $13^{24}$ , 4]
(3, 2)	54	$(2 + 2u)$	$\langle (x + \omega^2)^{53} \rangle$	$\langle (x - \omega^2)^{53} \rangle$	[54, $3^6$ , 27]
(5, 2)	50	4	$\langle u(x + 2)^4 \rangle$	$\langle u(x - 2)^{10} \rangle$	[50, $5^{72}$ , 4]
(17, 1)	34	8	$\langle (x + 12)^{13} \rangle$	$\langle (x - 12)^{13} + u(x - 12)^9 \rangle$	[34, $17^{16}$ , 30]
(13, 1)	338	10	$\langle (x + 7)^{137} + u, u(x + 7)^{131} \rangle$	$\langle (x - 7)^{158}, u(x - 7)^{156} \rangle$	[338, $13^{94}$ , 24]
(5, 3)	250	$\omega^6$	$\langle u(x + \omega^{65})^{123} \rangle$	$\langle (x - \omega^{65})^{124} \rangle$	[250, $5^{12}$ , 125]
(7, 2)	14	4	$\langle u(x + 5)^6 \rangle$	$\langle (x - 5)^6, u(x - 5)^5 \rangle$	[14, $7^5$ , 7]
(11, 2)	22	7	$\langle (x + \omega)^{102} + u(x + \omega^{102}), u(x + \omega^{102})^2 \rangle$	$\langle 0 \rangle$	[22, $11^{22}$ , 5]

**Table 4.**  $\Lambda$ -constacyclic codes over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ , for non-square  $\Lambda$ 

$(p, m)$	$n$	$\Lambda$	$\langle g(x) \rangle$	[ $n, M, d_{sp}$ ]
(3, 3)	18	$(2 + u)$	$\langle (x^2 - 2)^{17} \rangle$	[18, $3^6$ , 18]
(19, 1)	38	$(8 + 15u)$	$\langle (x^2 - 8)^{27} \rangle$	[38, $19^{22}$ , 10]
(23, 1)	46	$(21 + 8u)$	$\langle (x^2 - 21)^{21} \rangle$	[46, $23^{50}$ , 2]
(7, 1)	98	5	$\langle u(x^2 - 5)^{96} \rangle$	[98, $7^4$ , 84]
(11, 1)	242	10	$\langle (x^2 - 10)^{89} \rangle$	[242, $11^{64}$ , 20]
(7, 3)	98	3	$\langle (x^2 - 3)^{47} + u, u(x^2 - 3)^{46} \rangle$	[98, $7^{30}$ , 70]
(19, 1)	38	2	$\langle u(x^2 - 2)^{12} \rangle$	[38, $19^{14}$ , 26]
(3, 3)	162	2	$\langle (x^2 - 2)^1 + u \rangle$	[162, $3^{1344}$ , 8]
(11, 3)	22	6	$\langle (x^2 - 6)^{11}, u(x^2 - 6)^{10} \rangle$	[22, $11^6$ , 20]

### 4.3 MDS Symbol-Pair Codes over $\mathfrak{R}$

Here, by using the symbol-pair Singleton Bound over the chain ring  $\mathfrak{R}$ , the conditions for the existence of MDS symbol-pair codes are investigated.

In [1], Chee et al. obtained the symbol-pair Singleton bound over finite field  $\mathbb{F}_{p^m}$  as  $|\mathcal{C}| \leq p^{m(n-d_{sp}(\mathcal{C})+2)}$ . The following theorem generalizes the symbol-pair Singleton bound over finite chain rings.

**Corollary 4.3.1.** *(Symbol-pair Singleton Bound) [5] Consider a symbol-pair code  $\mathcal{C}$  of length  $n$  over  $\mathfrak{R}$  with symbol-pair distance  $d_{sp}(\mathcal{C})$ . The symbol-pair Singleton bound is given by*

$$|\mathcal{C}| \leq p^{2m(n-d_{sp}(\mathcal{C})+2)}.$$

**Definition 4.3.2.** Let  $\mathcal{C}$  be a symbol-pair code of length  $n$  over  $\mathfrak{R}$ . Then,  $\mathcal{C}$  is an MDS symbol-pair code if it attains the Singleton bound for symbol-pair codes.

Now, we explore all symbol-pair MDS codes of length  $2p^s$  for the case when  $\Lambda$  is non-square. First consider  $\Lambda$  is of the form  $(\Phi + u\Psi)$ .

**Theorem 4.3.3.** *Let  $\mathcal{C} = \langle (x^2 - \Phi_0)^j \rangle \subseteq \frac{\mathfrak{R}[x]}{\langle x^{2p^s} - (\Phi + u\Psi) \rangle}$  be a  $(\Phi + u\Psi)$ -constacyclic code of length  $2p^s$  over  $\mathfrak{R}$ . Then, the only MDS symbol-pair code is the ambient ring  $\frac{\mathfrak{R}[x]}{\langle x^{2p^s} - (\Phi + u\Psi) \rangle}$  itself.*

*Proof.* From Theorem 4.2.1, we have  $|\mathcal{C}| = p^{2m(2p^s-j)}$ .

**Case 1:** If  $0 \leq j \leq p^s$ , then  $d_{sp}(\mathcal{C}) = 2$  from Theorem 4.2.7.  $\mathcal{C}$  is MDS if and only if  $|\mathcal{C}| = p^{2m(2p^s-d_{sp}(\mathcal{C})+2)}$  i.e.,  $p^{2m(2p^s-j)} = p^{2m(2p^s-2+2)}$  i.e.,  $2p^s - j = 2p^s$  i.e.,  $j = 0$ . Hence,  $\mathcal{C} = \langle 1 \rangle$  is an MDS symbol-pair  $(\Phi + u\Psi)$ -constacyclic code of length  $2p^s$  over  $\mathfrak{R}$ .

**Case 2:** If  $2p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq j \leq 2p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ , then

$d_{sp}(\mathcal{C}) = 2(\wp+1)p^\sigma$ . So,  $\mathcal{C}$  is MDS if and only if  $|\mathcal{C}| = p^{2m(2p^s-d_{sp}(\mathcal{C})+2)}$  i.e.,  $p^{2m(2p^s-j)} = p^{2m(2p^s-d_{sp}(\mathcal{C})+2)}$  i.e.,  $2p^s - j = 2p^s - d_{sp}(\mathcal{C}) + 2$  i.e.,  $j = d_{sp}(\mathcal{C}) - 2$ . Now,

$$\begin{aligned}
j &\geq 2p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \\
&= p^s + p^{s-\sigma}(p^\sigma - 1) + (\wp - 1)p^{s-\sigma-1} + 1 \\
&\geq p^{\sigma+1} + p(p^\sigma - 1) + (\wp - 1) + 1 \text{ (equality when } \sigma = s - 1) \\
&\geq (\wp + 1)p^\sigma + (\wp + 1)(p^\sigma - 1) + \wp \text{ (equality when } \wp = p - 1) \\
&= 2(\wp + 1)p^\sigma - 1 \\
&> d_{sp}(\mathcal{C}) - 2.
\end{aligned}$$

Thus, no MDS symbol-pair code exists in this case.

**Case 3:** If  $j = 2p^s$ , then  $d_{sp}(\mathcal{C}) = 0$ . For  $\mathcal{C}$  to be MDS,  $|\mathcal{C}| = p^{2m(2p^s-d_{sp}(\mathcal{C})+2)}$  i.e.,  $1 = p^{2m(2p^s+2)}$  i.e.,  $p^s + 1 = 0$ , which is false for any  $s$  and  $p$ . Thus, there is no MDS symbol-pair code.  $\square$

Next, we take  $\Lambda$  to be of the form  $\Omega \in \mathbb{F}_{p^m}^*$ . First, we consider the Type 1  $\Omega$ -constacyclic codes symbol-pair of length  $2p^s$ .

**Theorem 4.3.4.** *Let  $\mathcal{C}$  be a Type 1  $\Omega$ -constacyclic symbol-pair code of length  $2p^s$  over  $\mathfrak{R}$ , then  $\langle 1 \rangle$  is the only MDS symbol-pair code.*

*Proof.* We have the following situations:

**Case 1:** When  $\mathcal{C} = \langle 0 \rangle$ , then  $d_{sp}(\mathcal{C}) = 0$ . For  $\mathcal{C}$  to be MDS we must have,  $|\mathcal{C}| = p^{2m(2p^s-d_{sp}(\mathcal{C})+2)}$  i.e.,  $1 = p^{2m(2p^s+2)}$  i.e.,  $p^s + 1 = 0$ , which is false for any  $s$  and  $p$ .

**Case 2:** When  $\mathcal{C} = \langle 1 \rangle$ , then  $d_{sp}(\mathcal{C}) = 2$ . For  $\mathcal{C}$  to be MDS,  $|\mathcal{C}| = p^{2m(2p^s-d_{sp}(\mathcal{C})+2)}$  i.e.,  $p^{4mp^s} = p^{2m(2p^s-2+2)}$ , which is true. Thus, the code  $\mathcal{C}$  is MDS.  $\square$

Now, we investigate the MDS condition for Type 2 symbol-pair  $\Omega$ -constacyclic

codes.

**Theorem 4.3.5.** *Let  $\mathcal{C} = \langle u(x^2 - \Omega_0)^j \rangle$ , be a symbol-pair  $\Omega$ -constacyclic code of Type 2 of length  $2p^s$  over  $\mathfrak{R}$ , where  $0 \leq j \leq p^s - 1$ . Then there is no MDS symbol-pair codes.*

*Proof. Case 1:* When  $j = 0$ , then  $d_{sp}(\mathcal{C}) = 2$ . For  $\mathcal{C}$  to be MDS,  $|\mathcal{C}| = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $p^{2mp^s} = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $p^{2mp^s} = p^{4mp^s}$ , which is false for any positive  $m, s$ , and  $p$ . Thus,  $\mathcal{C}$  is not MDS.

**Case 2:** When  $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq j \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ , then  $d_{sp}(\mathcal{C}) = 2(\wp + 1)p^\sigma$ .  $\mathcal{C}$  is an MDS code if and only if  $|\mathcal{C}| = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $p^{2m(p^s - j)} = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $j = d_{sp}(\mathcal{C}) - p^s - 2$ .

Now,

$$\begin{aligned}
j &\geq p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \\
&= p^s + p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 - p^s \\
&\geq p^{\sigma+1} + p^{s-\sigma}(p^\sigma - 1) + (\wp - 1) + 1 - p^s \text{ (equality when } \sigma = s - 1) \\
&\geq 2(\wp + 1)p^\sigma - p^s - 1 \text{ (equality when } p - 1 = \wp) \\
&> d_{sp}(\mathcal{C}) - p^s - 2.
\end{aligned}$$

Thus, there is no MDS symbol-pair code. □

Next we verify the MDS condition for Type 3 codes.

**Theorem 4.3.6.** *Let  $\mathcal{C} = \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$  be a symbol-pair  $\Omega$ -constacyclic code of length  $2p^s$  over  $\mathfrak{R}$ , where  $0 \leq \ell < j$ ,  $1 \leq j \leq p^s - 1$ , and either  $\mathfrak{S}(x)$  is 0 or a unit in  $\frac{\mathbb{F}_p^m[x]}{\langle x^{2p^s} - \Omega \rangle}$ . Then there exist MDS symbol-pair codes  $\mathcal{C}$  for all  $\chi$  satisfying any one of the following conditions:*

- When  $\mathfrak{S}(x) = 0$ 
  - When  $s = 1$ , then  $d_{sp}(\mathcal{C}) = 2(\chi + 1)$  for  $1 \leq \chi \leq p - 1$ .
  - When  $s \geq 2$ , then
    - $\chi = 1$ , then  $d_{sp}(\mathcal{C}) = 4$ ,
    - $\chi = p^s - 1$ , then  $d_{sp}(\mathcal{C}) = 2p^s$ .
- When  $\mathfrak{S}(x) \neq 0$ 
  - When  $s = 1$ , then  $d_{sp}(\mathcal{C}) = 2(\chi + 1)$  for  $1 \leq \chi \leq p - 1$ .
  - When  $s \geq 2$ , then
    - $\chi = 1$ , then  $d_{sp}(\mathcal{C}) = 4$ ,
    - $\chi = p^s - 1$ ,  $\ell = p^s - 2$ , then  $d_{sp}(\mathcal{C}) = 2p^s$ .

*Proof.* We have the following two cases:

**Case 1:** When  $\mathfrak{S}(x) = 0$  and  $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ , then  $d_{sp}(\mathcal{C}) = 2(\wp + 1)p^\sigma$ .  $\mathcal{C}$  is MDS if and only if  $|\mathcal{C}| = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $p^{4m(p^s - j)} = p^{2m(4p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $2j = d_{sp}(\mathcal{C}) - 2$  i.e.,  $2\chi = d_{sp}(\mathcal{C}) - 2$ .

Now,

$$\begin{aligned}
2\chi &\geq 2(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1) \\
&\geq 2(p^{\sigma+1} - p + (\wp - 1) + 1) \text{ (equality when } \sigma = s - 1) \\
&\geq 2(\wp + 1)p^\sigma - 2(\wp + 1) + 2(\wp - 1) + 2 \\
&\quad \text{(equality when } p - 1 = \wp) \\
&\geq d_{sp}(\mathcal{C}) - 2.
\end{aligned}$$

So,  $\mathcal{C}$  is an MDS symbol-pair constacyclic code if and only if  $s = 1$  (in such case,  $j = \wp$ ,  $d_{sp}(\mathcal{C}) = 2(\wp + 1)$ ), or  $\wp = 1$ ,  $\sigma = 0$  (in such case,  $j = 1$ ,  $d_{sp}(\mathcal{C}) = 4$ ), or  $\wp = p - 1$ ,  $\sigma = s - 1$  (in such case,  $j = p^s - 1$ ,  $d_{sp}(\mathcal{C}) = 2p^s$ ).

**Case 2:** When  $\mathfrak{S}(x) \neq 0$  and  $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \chi \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ , then we consider the following two subcases:

**Subcase 2.1:** When  $1 \leq j \leq \frac{p^s + \ell}{2}$ , then  $\chi = j$ .  $\mathcal{C}$  is MDS if and only if  $|\mathcal{C}| = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $p^{4m(p^s - j)} = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $2j = d_{sp}(\mathcal{C}) - 2$  i.e.,  $2\chi = d_{sp}(\mathcal{C}) - 2$ . Also, we have

$$\begin{aligned} 2\chi &\geq 2(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1) \\ &\geq 2p^{\sigma+1} - 2p + 2(\wp - 1) + 2 \text{ (equality when } \sigma = s - 1) \\ &\geq 2(\wp + 1)p^\sigma - 2(\wp + 1) + 2(\wp - 1) + 2 \\ &\quad \text{(equality when } p - 1 = \wp) \\ &= d_{sp}(\mathcal{C}) - 2. \end{aligned}$$

So,  $\mathcal{C}$  is an MDS symbol-pair constacyclic code if and only if  $s = 1$  (in such case,  $j = \wp$ ,  $d_{sp}(\mathcal{C}) = 2(\wp + 1)$ ), or  $\wp = 1$ ,  $\sigma = 0$  (in such case,  $j = 1$ ,  $d_{sp}(\mathcal{C}) = 4$ ), or  $\wp = p - 1$ ,  $\sigma = s - 1$  (in such case,  $j = p^s - 1$ ,  $\ell = p^s - 2$ ,  $d_{sp}(\mathcal{C}) = 2p^s$ ).

**Subcase 2.2:** When  $\frac{p^s + \ell}{2} < j \leq p^s - 1$  then  $\chi = p^s - j + \ell$ .  $\mathcal{C}$  is MDS if and only if  $|\mathcal{C}| = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $p^{2m(p^s - \ell)} = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$  i.e.,  $\ell = d_{sp}(\mathcal{C}) - p^s - 2$  i.e.,

$p^s + \ell = d_{sp}(\mathcal{C}) - 2$  i.e.,  $p^s - j + \ell = d_{sp}(\mathcal{C}) - j - 2$  i.e.,  $\chi = d_{sp}(\mathcal{C}) - j - 2$ . Now,

$$\begin{aligned}
\chi &\geq p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \\
&\geq p^{\sigma+1} - p + (\wp - 1) + 1 \text{ (equality when } \sigma = s - 1) \\
&\geq 2(\wp + 1)p^\sigma - p^s - (\wp + 1) + (\wp - 1) + 1 \\
&\quad \text{(equality when } p - 1 = \wp) \\
&\geq d_{sp}(\mathcal{C}) - (p^s - 1) - 2 \\
&\geq d_{sp}(\mathcal{C}) - j - 2 \text{ (equality when } j = p^s - 1) \\
&\geq d_{sp}(\mathcal{C}) - j - 2.
\end{aligned}$$

So,  $\mathcal{C}$  is an MDS symbol-pair constacyclic code if and only if  $\wp = p - 1$ ,  $\sigma = s - 1$  (in such case,  $j = p^s - 1$ ,  $\ell = p^s - 2$ ,  $d_{sp}(\mathcal{C}) = 2p^s$ ).  $\square$

Finally, we explore the MDS condition for Type 4  $\Omega$ -constacyclic codes.

**Theorem 4.3.7.** *Let  $\mathcal{C} = \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x), u(x^2 - \Omega_0)^\zeta \rangle$  be a symbol-pair  $\Omega$ -constacyclic code of Type 4 of length  $2p^s$  over  $\mathfrak{R}$ , where  $1 \leq j \leq p^s - 1$ ,  $0 \leq \ell < j$ , either  $\mathfrak{S}(x)$  is a unit in  $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{2p^s} - \Omega \rangle}$  or 0,  $\deg(\mathfrak{S}) \leq \zeta - \ell - 1$ ,  $\zeta < \chi$ , and  $\chi$  is the smallest possible integer such that  $u(x^2 - \Omega_0)^\chi \in \langle (x^2 - \Omega_0)^j + u(x^2 - \Omega_0)^\ell \mathfrak{S}(x) \rangle$ ; i.e.  $\chi = \min\{j, p^s - j + \ell\}$ , if  $\mathfrak{S}(x) \neq 0$ , otherwise  $\chi = j$ . Then, there is no MDS symbol-pair code.*

*Proof.* When  $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq \zeta \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$ , then the symbol-pair distance is  $d_{sp}(\mathcal{C}) = 2(\wp + 1)p^\sigma$ . So,  $\mathcal{C}$  is MDS if and only if  $|\mathcal{C}| = p^{2m(2p^s - d_{sp}(\mathcal{C}) + 2)}$

i.e.,  $p^{2m(2p^s-j-\zeta)} = p^{2m(2p^s-d_{sp}(C)+2)}$  i.e.,  $\zeta = d_{sp}(C) - j - 2$ . Now,

$$\begin{aligned}
\zeta &\geq p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \\
&\geq p^{\sigma+1} - p + (\wp - 1) + 1 \text{ (equality when } \sigma = s - 1) \\
&\geq 2(\wp + 1)p^\sigma - p^{\sigma+1} - (\wp + 1) + (\wp - 1) + 1 \\
&\text{(equality when } p - 1 = \wp) \\
&\geq d_{sp}(C) - p^s - 1 \\
&\geq d_{sp}(C) - j - 2 \\
&\text{(equality when } j = p^s - 1)
\end{aligned}$$

So, equality holds when  $\zeta = p^s - 1$ , which is impossible because  $\zeta < \chi \leq p^s - 1$ .

Hence, there is no maximum distance separable symbol-pair code in this case.  $\square$

**Example 4.3.8.** In Table 5, we provide some examples of MDS symbol-pair Type 3  $\Omega$ -constacyclic codes of length  $2p^s$  over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ , where  $\Omega \in \mathbb{F}_{p^m}^*$ . We fix a primitive element of  $\mathbb{F}_{p^m}$ , and we denote it by  $\omega$ .

**Table 5.** MDS  $\Omega$ -constacyclic codes of length  $2p^s$  over  $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

$(p, m)$	$n$	$\Omega$	$\langle g(x) \rangle$	$[n, M, d_{sp}]$
$(3, 2)$	54	$\omega$	$\langle (x^2 - \omega) \rangle$	$[54, 3^{208}, 4]$
$(5, 1)$	50	3	$\langle (x^2 - 3)^{24} \rangle$	$[50, 5^4, 50]$
$(7, 3)$	98	6	$\langle (x^2 - 6)^{48} + u(x^2 - 6)^{47} \rangle$	$[98, 7^6, 98]$
$(13, 2)$	26	$\omega^5$	$\langle (x^2 - \omega^{65})^{10} + u(x^2 - \omega^{65})^8 \rangle$	$[26, 13^{16}, 24]$
$(17, 1)$	34	7	$\langle (x^2 - 7)^8 \rangle$	$[34, 17^{36}, 18]$
$(11, 1)$	242	8	$\langle (x^2 - 8) \rangle$	$[242, 11^{480}, 4]$
$(3, 1)$	162	2	$\langle (x^2 - 2)^{80} + u(x^2 - 2)^{79} \rangle$	$[162, 3^{16}, 162]$
$(19, 1)$	38	12	$\langle (x^2 - 12)^{15} + u(x^2 - 12)^5 \rangle$	$[38, 19^{40}, 20]$
$(23, 1)$	46	11	$\langle (x^2 - 11)^{20} \rangle$	$[46, 23^{12}, 42]$

## 4.4 Conclusion

The distances of constacyclic codes have a very important role in error-correcting codes. The symbol-pair distances of all  $\Lambda$ -constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$  are thoroughly computed for each invertible element  $\Lambda$  of  $\mathfrak{R}$ . We explore all MDS symbol-pair constacyclic codes of length  $2p^s$  over  $\mathfrak{R}$ .

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