Chapter 3

On Symbol-Pair Distances of Constacyclic Codes of Length $4p^s$ over $\mathbb{F}_{p^m}$

3.1 Introduction

In this Chapter, we obtain the symbol-pair distances of λ-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$ for Cases 1, 2, and 3 of Table 1. Also we identify all symbol-pair MDS constacyclic codes of length $4p^s$ among them.

We organize the rest of the Chapter in the following manner. In Section 3.2, we give computational procedures to determine the symbol-pair distances of λ-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$ for Cases 1, 2, and 3 of Table 1. And in Section 3.3, we explore all symbol-pair MDS codes of length $4p^s$ among them and give some examples of MDS symbol-pair codes.

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3.2 Symbol-Pair Distances of Codes over $\mathbb{F}_{p^m}$

Let $\mathbb{F}_{p^m}$ be the finite field, where p is an odd prime and m is a positive integer such that $p^m \equiv 1 \pmod 4$. As discussed in [2], λ-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$ are ideals of the ring $R_\lambda = \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s} - \lambda) \rangle}$. Let $\lambda = \gamma^{p^s}$ where γ is a positive element of $\mathbb{F}_{p^m}$. It is a well known fact that the ring R_λ is a principal ideal ring, whose ideals are generated by the factors of $x^{4p^s} - \lambda = (x^4 - \gamma)^{p^s}$. In [2], Dinh classified all constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$. We list all those structures of constacyclic codes of length $4p^s$ for $p^m \equiv 1 \pmod 4$ over $\mathbb{F}_{p^m}$ in Table 1 as follows:

Table 1. List of λ-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$

(when $p^m \equiv 1 \pmod 4$)

$Cases$	λ	$C = \langle g(x) \rangle$	$\lvert C \rvert$
$Case\ 1$	$\varepsilon\xi_1^4$	$\langle (x^4 - \varepsilon^{p^{m-\nu}})^j \rangle$	$p^{m(4p^s-4j)}$
$Case\ 2$	$\varepsilon^3\xi_2^4$	$\langle (x^4 - \varepsilon^{3p^{m-\nu}})^j \rangle$	$p^{m(4p^s-4j)}$
$Case\ 3$	$\varepsilon^2\xi_3^4$	$\langle (x^2 - \varepsilon^{p^{m-\nu}})^i (x^2 + \varepsilon^{p^{m-\nu}})^j \rangle$	$p^{m(4p^s-2i-2j)}$
$Case\ 4$	ξ_4^4	$\langle (x+1)^j (x-1)^i (x+\rho)^l (x-\rho)^k \rangle$	$p^{m(4p^s-i-j-k-l)}$

where $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathbb{F}_{p^m}$, $\rho^2 = -1$, $s \equiv \nu \pmod m$, $0 \le i, j, k, l \le p^s$

and ε is a primitive element of $\mathbb{F}_{p^m}$.

The Hamming distances of Cases 1, 2, 3, and 4 have been computed in [5, 7]. Here, we list all those Hamming distances in simplified form as follows:

Theorem 3.2.1. *[5, 7] Let $C = \langle (x^4 - \gamma)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s} - \lambda) \rangle}$, for $0 \le j \le p^s$ be a λ-constacyclic code as in Case 1 or Case 2 of Table 1. Then the Hamming distance distribution $d_H(C)$ is completely given by*

- $d_H(C) = 1$, *if* $j = 0$

- $d_H(C) = (\wp + 1)p^{\sigma}$, *if* $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq j \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$
- $d_H(C) = 0$, *if* $j = p^s$

where $0 \leq \sigma \leq s-1$, $1 \leq \wp \leq p-1$.

Theorem 3.2.2. *[5, 7] Let* $C = \langle (x^2-\gamma_1)^i (x^2+\gamma_1)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s}-\lambda) \rangle}$, *for* $0 \leq i \leq j \leq p^s$ *be a* λ*-constacyclic code as in Case* 3 *of Table* 1. *Then the Hamming distances,* $d_H(C)$ *are completely computed as*

- $d_H(C) = 1$, *if* $i = j = 0$
- $d_H(C) = 2$, *if* $i = 0$ *and* $0 < j \leq p^s$
- $d_H(C) = \min\{(\wp_1 + 1)p^{\sigma_1}, 2(\wp_2 + 1)p^{\sigma_2}\}$, *if* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ *and* $p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s - p^{s-\sigma_2} + \wp_2 p^{s-\sigma_2-1}$
- $d_H(C) = 2(\wp_1 + 1)p^{\sigma_1}$, *if* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ *and* $j = p^s$
- $d_H(C) = 0$, *if* $i = j = p^s$

where $0 \leq \sigma_1 \leq \sigma_2 \leq s-1$, $1 \leq \wp_1, \wp_2 \leq p-2$.

Remark 3.2.3. [5, 7] The case with $i \geq j$ has the same Hamming distances as Theorem 3.2.2 by symmetries.

Theorem 3.2.4. *[5] Let* $C = \langle (x+1)^i (x-1)^j (x+\rho)^k (x-\rho)^l \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s}-\lambda) \rangle}$ *be a* λ*-constacyclic code as in Case* 4 *of Table* 1, *for* $0 \leq l \leq j \leq k \leq i \leq p^s$. *Then the Hamming distances,* $d_H(C)$ *are completely computed as*

- $d_H(C) = 1$, *if* $i = j = k = l = 0$

- $d_H(C) = 2$, *if* $j = k = l = 0$ *and* $0 < i \leq p^s$, *or* $l = 0$ *and* $0 < j \leq k \leq i \leq p^{s-1}$
- $d_H(C) = 3$, *if* $j = l = 0$, $0 < k \leq p^s$ *and* $p^{s-1} < i \leq p^s$, *or* $l = 0$, $0 < j \leq k \leq 2p^{s-1}$ *and* $p^{s-1} < i \leq 2p^{s-1}$
- $d_H(C) = 4$, *if* $l = 0$, $0 < j \leq k \leq p^s$ *and* $2p^{s-1} < i \leq p^s$
- $d_H(C) = \min\{(\wp_1+1)p^{\sigma_1}, 2(\wp_2+1)p^{\sigma_2}, 3(\wp_3+1)p^{\sigma_3}, 4(\wp_4+1)p^{\sigma_4}\}$, *if* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$, $p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s - p^{s-\sigma_2} + \wp_2 p^{s-\sigma_2-1}$ $p^s - p^{s-\sigma_3} + (\wp_3 - 1)p^{s-\sigma_3-1} + 1 \leq k \leq p^s - p^{s-\sigma_3} + \wp_3 p^{s-\sigma_3-1}$ *and* $p^s - p^{s-\sigma_4} + (\wp_4 - 1)p^{s-\sigma_4-1} + 1 \leq l \leq p^s - p^{s-\sigma_4} + \wp_4 p^{s-\sigma_4-1}$
- $d_H(C) = \min\{2(\wp_2+1)p^{\sigma_2}, 3(\wp_3+1)p^{\sigma_3}, 4(\wp_4+1)p^{\sigma_4}\}$, *if* $i = p^s$, $p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s - p^{s-\sigma_2} + \wp_2 p^{s-\sigma_2-1}$ $p^s - p^{s-\sigma_3} + (\wp_3 - 1)p^{s-\sigma_3-1} + 1 \leq k \leq p^s - p^{s-\sigma_3} + \wp_3 p^{s-\sigma_3-1}$ *and* $p^s - p^{s-\sigma_4} + (\wp_4 - 1)p^{s-\sigma_4-1} + 1 \leq l \leq p^s - p^{s-\sigma_4} + \wp_4 p^{s-\sigma_4-1}$
- $d_H(C) = \min\{3(\wp_3+1)p^{\sigma_3}, 4(\wp_4+1)p^{\sigma_4}\}$, *if* $i = j = p^s$, $p^s - p^{s-\sigma_3} + (\wp_3 - 1)p^{s-\sigma_3-1} + 1 \leq k \leq p^s - p^{s-\sigma_3} + \wp_3 p^{s-\sigma_3-1}$ *and* $p^s - p^{s-\sigma_4} + (\wp_4 - 1)p^{s-\sigma_4-1} + 1 \leq l \leq p^s - p^{s-\sigma_4} + \wp_4 p^{s-\sigma_4-1}$
- $d_H(C) = 4(\wp_4+1)p^{\sigma_4}$, *if* $i = j = k = p^s$ *and* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq l \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$
- $d_H(C) = 0$, *if* $i = j = k = l = p^s$

where $0 \leq \sigma_4 \leq \sigma_3 \leq \sigma_2 \leq \sigma_1 \leq s-1$, $1 \leq \wp_1, \wp_2, \wp_3, \wp_4 \leq p-2$.

Remark 3.2.5. [5] The case with $j \leq l \leq k \leq i, k \leq j \leq l \leq i, j \leq k \leq l \leq i, l \leq i \leq k \leq j, k \leq i \leq l \leq j, i \leq k \leq l \leq j, i \leq l \leq k \leq j, l \leq j \leq i \leq k, l \leq i \leq j \leq$

$k, i \leq j \leq j \leq k, j \leq l \leq i \leq k, i \leq k \leq j \leq l, k \leq i \leq j \leq l, k \leq j \leq i \leq l$ and $j \leq k \leq i \leq l$ have the same Hamming distances as Theorem 3.2.4 by symmetries.

Theorem 3.2.6. *[5] Let* $C = \langle (x+1)^i (x-1)^j (x+\rho)^k (x-\rho)^l \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s} - \lambda) \rangle}$ *be a* λ*-constacyclic code as in Case* 4 *of Table* 1*, for* $0 \leq l \leq k \leq j \leq i \leq p^s$*. Then the Hamming distances,* $d_H(C)$ *are completely computed as*

- $d_H(C) = 1$*, if* $i = j = k = l = 0$
- $d_H(C) = 2$*, if* $k = l = 0$ *and* $0 \leq j \leq i \leq p^s$*, or* $l = 0$ *(but not* $i = j = 0$*), or* $l = 0$ *and* $0 < k \leq j \leq i \leq p^{s-1}$
- $d_H(C) = 4$*, if* $l = 0$*,* $0 < k \leq j \leq p^s$ *and* $p^{s-1} < i \leq p^s$
- $d_H(C) = \min\{(\wp_1 + 1)p^{\sigma_1}, 2(\wp_3 + 1)p^{\sigma_3}, 4(\wp_4 + 1)p^{\sigma_4}\}$*, if* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$*,* $p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s - p^{s-\sigma_2} + \wp_2 p^{s-\sigma_2-1}$ $p^s - p^{s-\sigma_3} + (\wp_3 - 1)p^{s-\sigma_3-1} + 1 \leq k \leq p^s - p^{s-\sigma_3} + \wp_3 p^{s-\sigma_3-1}$ *and* $p^s - p^{s-\sigma_4} + (\wp_4 - 1)p^{s-\sigma_4-1} + 1 \leq l \leq p^s - p^{s-\sigma_4} + \wp_4 p^{s-\sigma_4-1}$
- $d_H(C) = \min\{2(\wp_3 + 1)p^{\sigma_3}, 4(\wp_4 + 1)p^{\sigma_4}\}$*, if* $i = p^s$*,* $p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s$ $p^s - p^{s-\sigma_3} + (\wp_3 - 1)p^{s-\sigma_3-1} + 1 \leq k \leq p^s - p^{s-\sigma_3} + \wp_3 p^{s-\sigma_3-1}$ *and* $p^s - p^{s-\sigma_4} + (\wp_4 - 1)p^{s-\sigma_4-1} + 1 \leq l \leq p^s - p^{s-\sigma_4} + \wp_4 p^{s-\sigma_4-1}$
- $d_H(C) = 4(\wp_4 + 1)p^{\sigma_4}$*, if* $i = j = k = p^s$ *and* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq l \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$
- $d_H(C) = 0$*, if* $i = j = k = l = p^s$

where $0 \leq \sigma_4 \leq \sigma_3 \leq \sigma_2 \leq \sigma_1 \leq s - 1$*,* $1 \leq \wp_1, \wp_2, \wp_3, \wp_4 \leq p - 2$*.*

Remark 3.2.7. [5] The case with $k \leq l \leq j \leq i, l \leq k \leq i \leq j, k \leq l \leq i \leq j, j \leq i \leq l \leq k, i \leq j \leq l \leq k, i \leq j \leq k \leq l$ and $j \leq i \leq k \leq l$ have the same Hamming distances as Theorem 3.2.6 by symmetries.

In this chapter, we aim to establish the symbol-pair distances of λ-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$ for the Case 1, Case 2, and Case 3 of Table 1.

Using the concept of the coefficient weight, $\mathrm{cw}(\phi(x))$ of a polynomial $\phi(x)$, we will determine the symbol-pair distances.

Definition 3.2.8. [3] Consider a polynomial with degree n, $\phi(x) = \phi_0 + \phi_1 x + ... + \phi_n x^n$. The coefficient weight of $\phi(x)$ is defined as

- $\mathrm{cw}(\phi(x)) = 0$, if $\phi(x)$ is a monomial
- $\mathrm{cw}(\phi(x)) = \min\{|i-j| : i \neq j, \phi_i \neq 0, \phi_j \neq 0\}$, otherwise.

The following result give the symbol-pair weight of product of two polynomials.

Lemma 3.2.9. *[4] Suppose $\phi(x), \psi(x)$ are two nonzero polynomials, and $\phi(x)\psi(x)$ represents a codeword of length n. If $\deg(\phi(x)) + \deg(\psi(x)) \leq n-2$, and $0 \leq \deg(\psi(x)) \leq \mathrm{cw}(\phi(x)) - 2$, then $\mathrm{wt}_{sp}(\phi(x)\psi(x)) = \mathrm{wt}_H(\phi(x)) \cdot \mathrm{wt}_{sp}(\psi(x))$.*

Now, we first consider the Case 1 and Case 2.

3.2.1 Case 1 and Case 2

From Table 1, in Case 1 and Case 2 the polynomial $(x^4 - \gamma)$ is irreducible in $\mathbb{F}_{p^m}[x]$. Therefore it is clear that the codes in Case 1 and Case 2 have identical symbol-pair distance distributions. That means, if we want to obtain the symbol-pair distances of codes in Case 1 and Case 2, we only need to determine the symbol-pair distances of the codes in Case 1 or Case 2.

Consider the λ-constacyclic codes $C = \langle (x^4-\gamma)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s}-\lambda) \rangle}$ of length $4p^s$ over $\mathbb{F}_{p^m}$, for $0 \leq j \leq p^s$. It is well known that the ring R_λ is a chain ring with the maximal ideal generated by $(x^4-\gamma)$ with nilpotency index p^s. Then we have, $R_\lambda = \langle (x^4-\gamma)^0 \rangle \supset \langle (x^4-\gamma)^1 \rangle \supset \langle (x^4-\gamma)^2 \rangle \supset \cdots \supset \langle (x^4-\gamma)^{p^s-1} \rangle \supset \langle (x^4-\gamma)^{p^s} \rangle = \langle 0 \rangle$. Obviously, $d_{sp}(\langle (x^4-\gamma)^0 \rangle) = 2$ and $d_{sp}(\langle (x^4-\gamma)^{p^s} \rangle) = 0$. Now, we consider C for $1 \leq j \leq p^s-1$.

Consider an arbitrary nonzero element $e(x) \in C$. Then there exists a nonzero element $\phi(x) \in R_\lambda$ such that $e(x) = (x^4-\gamma)^k \phi(x)$. By the Division Algorithm, we can assume that $\deg(\phi(x)) \leq 4p^s-4j-1$. Let $\phi(x) = \phi_0+\phi_1 x+\cdots+\phi_k x^k$; where $\phi_0, \phi_1, ..., \phi_k \in \mathbb{F}_{p^m}$ and $k = 4p^s-4j-1$. We divide $\phi(x)$ into two parts, namely, $\phi_{odd}(x)$ and $\phi_{even}(x)$ where $\phi_{odd}(x)$ contains only odd powers of x and $\phi_{even}(x)$ has only even powers of x. We consider three cases.

Case 1: $\phi_{odd}(x) = 0$ i.e. $\phi(x) = \phi_{even}(x)$.

Then

$$\begin{aligned} \mathrm{wt}_H(e(x)) &= \mathrm{wt}_H((x^4-\gamma)^j \phi(x)) \\ &= \mathrm{wt}_H \left[\left(\sum_{l=0}^{j} \binom{j}{l} (-\gamma)^{j-l} x^{4l} \right) \phi_{even}(x) \right]. \end{aligned}$$

Therefore, the nonzero terms of $e(x)$ are $4l+2$ positions apart for $l = 0, 1, ..., p^s-1$. Hence, we get $\mathrm{wt}_{sp}(e(x)) = 2\,\mathrm{wt}_H(e(x)) \geq 2d_H(C)$.

Case 2: $\phi_{even}(x) = 0$ i.e. $\phi(x) = \phi_{odd}(x)$.

Then

$$\mathrm{wt}_H(e(x)) = \mathrm{wt}_H((x^4-\gamma)^j \phi(x))$$

$$= \mathrm{wt}_H\left[\left(\sum_{l=0}^{j}\binom{j}{l}(-\gamma)^{j-l}x^{4l}\right)\phi_{odd}(x)\right].$$

Therefore, the nonzero terms of $e(x)$ are $4l+1$ positions apart for $l = 0, 1, ..., p^s - 1$. Hence, we get $\mathrm{wt}_{sp}(e(x)) = 2\,\mathrm{wt}_H(e(x)) \geq 2d_H(C)$.

Case 3: $\phi_{odd}(x) \neq 0$ and $\phi_{even}(x) \neq 0$.

Then

$$(x^4-\gamma)^j\phi(x) = (x^4-\gamma)^j\phi_{odd}(x) + (x^4-\gamma)^j\phi_{even}(x)$$

Clearly, $\mathrm{wt}_H((x^4-\gamma)^j\phi(x)) = \mathrm{wt}_H((x^4-\gamma)^j\phi_{odd}(x)) + \mathrm{wt}_H((x^4-\gamma)^j\phi_{even}(x))$.
Since, $(x^4-\gamma)^j\phi_{odd}(x)$ and $(x^4-\gamma)^j\phi_{even}(x)$ both are nonzero elements of C therefore, $\mathrm{wt}_H((x^4-\gamma)^j\phi_{odd}(x)) \geq d_H(C)$ and $\mathrm{wt}_H((x^4-\gamma)^j\phi_{even}(x)) \geq d_H(C)$.
Hence, $\mathrm{wt}_{sp}(e(x)) \geq \mathrm{wt}_H((x^4-\gamma)^j\phi(x)) \geq 2d_H(C)$.
Therefore for any $e(x) \in C$, we have $\mathrm{wt}_{sp}(e(x)) \geq 2d_H(C)$. So, we have $d_{sp}(C) \geq 2d_H(C)$. But, we know that $d_{sp}(C) \leq 2d_H(C)$, which follows that $d_{sp}(C) = 2d_H(C)$. So, using the Hamming distances given in Theorem 3.2.1, we can determine the symbol-pair distance of C as follows:

Theorem 3.2.10. *Let* $C = \langle(x^4-\gamma)^j\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle(x^{4p^s}-\lambda)\rangle}$ *be a* λ*-constacyclic code as in Case* 1 *or Case* 2 *of Table* 1*, for* $0 \leq j \leq p^s$*. Then the symbol-pair distance distribution* $d_{sp}(C)$ *is completely computed by*

- $d_{sp}(C) = 2$*, if* $j = 0$
- $d_{sp}(C) = 2(\wp+1)p^{\sigma}$*, if* $p^s - p^{s-\sigma} + (\wp-1)p^{s-\sigma-1} + 1 \leq j \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$
- $d_{sp}(C) = 0$*,* $j = p^s$

where $0 \leq \sigma \leq s-1$*,* $1 \leq \wp \leq p-1$*.*

Next, we consider the symbol-pair distances of λ-constacyclic codes for Case 3 in Table 1.

3.2.2 Case 3

In Case 3 of Table 1, the factorization of the polynomial $(x^4-\gamma)$ into irreducible factors in $\mathbb{F}_{p^m}[x]$ is $(x^4-\gamma)=(x^2-\gamma_1)(x^2+\gamma_1)$. Consider the λ-constacyclic codes, $C=\langle (x^2-\gamma_1)^i(x^2+\gamma_1)^j\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s}-\lambda)\rangle}$ of length $4p^s$ over $\mathbb{F}_{p^m}$, for $0\leq i\leq j\leq p^s$.

If $i=j$ then clearly, $C=\langle (x^2-\gamma_1)^j(x^2+\gamma_1)^j\rangle = \langle (x^4-\gamma)^j\rangle$. Using the same argument as the above Subsection 3.2.1, we can compute the symbol-pair distance of C as follows:

Proposition 3.2.11. *Let* $C=\langle (x^4-\gamma)^j\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s}-\lambda)\rangle}$ *be a* λ*-constacyclic code, for* $0\leq j\leq p^s$*. Then the symbol-pair distance distribution* $d_{sp}(C)$ *is completely computed by*

- $d_{sp}(C)=2$, *if* $j=0$
- $d_{sp}(C)=2(\wp+1)p^{\sigma}$, *if* $p^s-p^{s-\sigma}+(\wp-1)p^{s-\sigma-1}+1\leq j\leq p^s-p^{s-\sigma}+\wp p^{s-\sigma-1}$
- $d_{sp}(C)=0$, *if* $j=p^s$

where $0\leq \sigma\leq s-1$, $1\leq \wp\leq p-1$.

Now, we consider the symbol-pair distance of C for $i<j$.

Proposition 3.2.12. *Let* $i<j$ *be such that* $i=0$ *or* p^s-1 *and* $1\leq j\leq p^s$ *. Then the symbol-pair distances of* $C=\langle (x^2-\gamma_1)^i(x^2+\gamma_1)^j\rangle$ *are determined as follows:*

- $d_{sp}(C)=4$, *if* $i=0, 1\leq j\leq p^s$

- $d_{sp}(C) = 4p^s$, *if* $i = p^s - 1, j = p^s$

Proof. There are only two possible cases.

Case 1: If $i = 0, 1 \leq j \leq p^s$, then clearly $R_\lambda \supset \langle (x^2+\gamma_1)^1) \rangle \supset \langle (x^2+\gamma_1)^2) \rangle \supset \cdots \supset \langle (x^2+\gamma_1)^{p^s-1}) \rangle \supset \langle (x^2+\gamma_1)^{p^s}) \rangle$. Therefore, we have $d_{sp}(\langle (x^2+\gamma_1)^1) \rangle) \leq d_{sp}(\langle (x^2+\gamma_1)^2) \rangle) \leq \cdots \leq d_{sp}(\langle (x^2+\gamma_1)^{p^s-1}) \rangle) \leq d_{sp}(\langle (x^2+\gamma_1)^{p^s}) \rangle)$. Assume $C = \langle (x^2+\gamma_1)^1) \rangle$ has a codeword of symbol-pair weight 3. Then there exists an integer $h \in \{2, 3, ..., p^s\}$ and nonzero elements $h_1, h_2 \in \mathbb{F}_{p^m}$ such that $h_1 x^h + h_2 x^{h+1} \in C = \langle (x^2+\gamma_1)^1) \rangle$. Clearly, this is not possible since $C = \langle (x^2+\gamma_1)^1) \rangle$. Hence, $d_{sp}(\langle (x^2+\gamma_1)^1) \rangle) \geq 4$. Now, $x^{2p^s} + \lambda_1^{p^s} = (x^2+\gamma_1)^{p^s} \in C = \langle (x^2+\gamma_1)^{p^s}) \rangle$ and $\mathrm{wt}_{sp}(x^{2p^s} + \lambda_1^{p^s}) = 4$. Thus $d_{sp}(\langle (x^2+\gamma_1)^{p^s}) \rangle) \leq 4$. So, we get $4 \leq d_{sp}(\langle (x^2+\gamma_1)^1) \rangle) \leq d_{sp}(\langle (x^2+\gamma_1)^2) \rangle) \leq \cdots \leq d_{sp}(\langle (x^2+\gamma_1)^{p^s-1}) \rangle) \leq d_{sp}(\langle (x^2+\gamma_1)^{p^s}) \rangle) \leq 4$, forcing $d_{sp}(\langle (x^2+\gamma_1)^j) \rangle) = 4$ for $1 \leq j \leq p^s$.

Case 2: If $i = p^s - 1, j = p^s$, then clearly, $C = \langle (x^2-\gamma_1)^{p^s-1}(x^2+\gamma_1)^{p^s} \rangle$ contains codewords of the form $z(x^4-\gamma)^{p^s-1}(x^2+\gamma_1), z \in \mathbb{F}_{p^m}$. As, $z(x^4-\gamma)^{p^s-1}(x^2+\gamma_1) = \sum\limits_{l=0}^{p^s-1} z(-\gamma_1)^{p^s-1-l} x^{4l} + \sum\limits_{l=0}^{p^s-1} z(-\gamma_1)^{p^s-1-l} x^{4l+2}$ having symbol-pair weight $4p^s$, it follows that $d_{sp}(\langle (x^2-\gamma_1)^{p^s-1}(x^2+\gamma_1)^{p^s} \rangle) = 4p^s$. □

From Theorem 3.1 of [6], we have the following proposition, which shows that the symbol-pair weight of $q(x)(x^2+\gamma_1)^j \pmod{x^{2p^s}+\gamma} \in \dfrac{\mathbb{F}_{p^m}[x]}{\langle x^{2p^s} - \gamma \rangle}$ is less than or equal to the symbol-pair weight of the polynomial $q(x)(x^2+\gamma_1)^j \in \dfrac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s} - \lambda) \rangle}$.

Proposition 3.2.13. *[6] Let* $e(x) = q(x)(x^2+\gamma_1)^j \pmod{x^{2p^s}+\gamma}$*, where* $q(x)$ *is a nonzero polynomial over* $\mathbb{F}_{p^m}[x]$*, then* $e(x)$ *can represent a codeword of length* $2p^s$ *over* $\mathbb{F}_{p^m}$ *and the symbol-pair weight* wt_{sp} *satisfies*

- $\mathrm{wt}_{sp}(e(x)) \geq 0$, *if* $j = 0$

- $\mathrm{wt}_{sp}(e(x)) \geq 2(\wp+1)p^{\sigma}$, *if* $p^s-p^{s-\sigma}+(\wp-1)p^{s-\sigma-1}+1 \leq j \leq p^s-p^{s-\sigma}+\wp p^{s-\sigma-1}$

where $0 \leq \sigma \leq s-1$, $1 \leq \wp \leq p-1$.

Next, we consider the symbol-pair distance when $i \neq j$ and $i < j$.

Proposition 3.2.14. *Let* $i < j$ *be such that* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ *and* $p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s - p^{s-\sigma_2} + \wp_2 p^{s-\sigma_2-1}$. *Then the symbol-pair distances of* $C = \langle (x^2-\gamma_1)^i(x^2+\gamma_1)^j \rangle$ *are determined as follows:*

$$d_{sp}(C) = \min\{4(\wp_1+1)p^{\sigma_1}, 2(\wp_2+1)p^{\sigma_2}\},$$

where $0 \leq \sigma_1 \leq \sigma_2 \leq s-1$, $1 \leq \wp_1, \wp_2 \leq p-2$.

Proof. Consider an arbitrary nonzero element $e(x) \in C$. Then there exists a nonzero polynomial $\phi(x) \in R_\lambda$ such that $e(x) = \phi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^j$. Suppose that t is the largest positive integer with $(x^2-\gamma_1)^t|\phi(x)$ and $\phi(x) = \psi(x)(x^2-\gamma_1)^t$. Now, we have the following cases.

Case 1: $j + t < p^s$. Let $q(x) = \phi(x)(x^2-\gamma)^i$, then $e(x) = q(x)(x^2+\gamma_1)^j$. By the Division Algorithm, we can assume $\deg(e(x)) = n \leq 4p^s - 1$. Let us represent $e(x)$ as $e(x) = e_0 + e_1x + ... + e_nx^n + ... + e_{4p^s-2}x^{4p^s-2} + e_{4p^s-1}x^{4p^s-1}$, where $e_{n+1} = e_{n+2} = ... = e_{4p^s-1} = 0$. Now, we have $e(x) \pmod{x^{2p^s}+\gamma} = (e_0 - \gamma e_{2p^s}) + (e_1 - \gamma e_{2p^s+1})x + ... + (e_{2p^s-2} - \gamma e_{4p^s-2})x^{2p^s-2} + (e_{2p^s-1} - \gamma e_{4p^s-1})x^{2p^s-1}$. Also, $\mathrm{wt}_{sp}(e(x)) \geq \mathrm{wt}_{sp}(e(x) \pmod{x^{2p^s}+\gamma}$ and $e(x) \pmod{x^{2p^s}+\gamma}$ can be seen as a codeword of length $2p^s$ over $\mathbb{F}_{p^m}$. Using Proposition 3.2.13, we get $\mathrm{wt}_{sp}(e(x)) \geq \mathrm{wt}_{sp}(e(x) \pmod{x^{2p^s}+\gamma} \geq 2(\wp_2+1)p^{\sigma_2}$.

Case 2: $j+t \geq p^s$. Let t_1 be the nonnegative integer such that $j+t = p^s + t_1$. By the Division Algorithm, we can assume $\deg(\psi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^{t_1}) \leq 4p^s - 1$. Then we have,

$$\begin{aligned}
\mathrm{wt}_{sp}(e(x)) &= \mathrm{wt}_{sp}(\psi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^{p^s+t_1}) \\
&= \mathrm{wt}_{sp}(x^{2p^s}\psi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^{t_1}) \\
&+ \mathrm{wt}_{sp}(\gamma_1^{p^s}\psi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^{t_1}) \\
&= 2\,\mathrm{wt}_{sp}(\psi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^{t_1})
\end{aligned}$$

Clearly, $\psi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^{t_1}$ can be seen as a codeword of length $2p^s$ over $\mathbb{F}_{p^m}$. Let $q_1(x) = \psi(x)(x^2+\gamma_1)^{t_1}$. Then from Proposition 3.2.13 we have, $\mathrm{wt}_{sp}(q_1(x)(x^2-\gamma_1)^i) \geq 2(\wp_1+1)p^{\sigma_1}$. It follows that, $\mathrm{wt}_{sp}(e(x)) = 4(\wp_1+1)p^{\sigma_1}$.

Now, combining both the cases, we have, $d_{sp}(C) \geq \min\{4(\wp_1+1)p^{\sigma_1}, 2(\wp_2+1)p^{\sigma_2}\}$.

Now, consider the polynomial $(x^4-\gamma)^{p^s-p^{s-\sigma_2}+\wp_2 p^{s-\sigma_2-1}} \in C$ and we have,

$$\begin{aligned}
&(x^4-\gamma)^{p^s-p^{s-\sigma_2}+\wp_2 p^{s-\sigma_2-1}} \\
&= \left((x^4-\gamma)^{p^{s-\sigma_2}}\right)^{p^{\sigma_2}-1}\left((x^4-\gamma)^{p^{s-\sigma_2-1}}\right)^{\wp_2} \\
&= \left((x^{4p^{s-\sigma_2}}-\gamma')\right)^{p^{\sigma_2}-1}\left((x^{4p^{s-\sigma_2-1}}-\gamma'')\right)^{\wp_2} \\
&= \left[\left(\sum_{l=0}^{p^{\sigma_2}-1}\binom{p^{\sigma_2}-1}{l}(-\gamma')^{p^{\sigma_2}-1-l}x^{4p^{s-\sigma_2}l}\right)\right]\left[\left(\sum_{l=0}^{\wp_2}\binom{\wp_2}{l}(-\gamma'')^{\wp_2-l}x^{4p^{s-\sigma_2-1}l}\right)\right]
\end{aligned}$$

where $\gamma' = \gamma^{p^{s-\sigma_2}}, \gamma'' = \gamma^{p^{s-\sigma_2-1}}$. So we get,

$$\begin{aligned}
&\operatorname{cw}\left(\sum_{l=0}^{p^{\sigma_2}-1}\binom{p^{\sigma_2}-1}{l}(-\gamma')^{p^{\sigma_2}-1-l}x^{4p^{s-\sigma_2}l}\right)=4p^{s-\sigma_2},\\
&\mathrm{wt}_H\left(\sum_{l=0}^{p^{\sigma_2}-1}\binom{p^{\sigma_2}-1}{l}(-\gamma')^{p^{\sigma_2}-1-l}x^{4p^{s-\sigma_2}l}\right)=p^{\sigma_2},\\
&\deg\left(\sum_{l=0}^{\wp_2}\binom{\wp_2}{l}(-\gamma'')^{\wp_2-l}x^{4p^{s-\sigma_2-1}l}\right)=4\wp_2p^{s-\sigma_2-1}<4p^{s-\sigma_2}\\
&\text{and}\quad \mathrm{wt}_H\left(\sum_{l=0}^{\wp_2}\binom{\wp_2}{l}(-\gamma'')^{\wp_2-l}x^{4p^{s-\sigma_2-1}l}\right)=\wp_2+1.
\end{aligned}$$

Now,

$$\begin{aligned}
&\mathrm{wt}_H\left((x^4-\gamma)^{p^s-p^{s-\sigma_2}+\wp_2p^{s-\sigma_2-1}}\right)\\
&=\mathrm{wt}_H\left(\sum_{l=0}^{p^{\sigma_2}-1}\binom{p^{\sigma_2}-1}{l}(-\gamma')^{p^{\sigma_2}-1-l}x^{4p^{s-\sigma_2}l}\right)\mathrm{wt}_H\left(\sum_{l=0}^{\wp_2}\binom{\wp_2}{l}(-\gamma'')^{\wp_2-l}x^{4p^{s-\sigma_2-1}l}\right)\\
&=(\wp_2+1)p^{\sigma_2}.
\end{aligned}$$

Since any term of $(x^4-\gamma)^{p^s-p^{s-\sigma_2}+\wp_2p^{s-\sigma_2-1}}$ has power $4l$ of x, therefore

$$\begin{aligned}
&\mathrm{wt}_{sp}\left((x^4-\gamma)^{p^s-p^{s-\sigma_2}+\wp_2p^{s-\sigma_2-1}}\right)\\
&=2\,\mathrm{wt}_H\left((x^4-\gamma)^{p^s-p^{s-\sigma_2}+\wp_2p^{s-\sigma_2-1}}\right)\\
&=2(\wp_2+1)p^{\sigma_2}.
\end{aligned}$$

Again consider the polynomial $(x^2-\gamma)^{p^s-p^{s-\sigma_1}+\wp_1p^{s-\sigma_1-1}}(x^2+\gamma_1)^{p^s}\in C$, proceeding as above we get $\mathrm{wt}_{sp}((x^2-\gamma)^{p^s-p^{s-\sigma_1}+\wp_1p^{s-\sigma_1-1}}(x^2+\gamma_1)^{p^s})=4(\wp_1+1)p^{\sigma_1}$. So, $d_{sp}(C)\leq\min\{4(\wp_1+1)p^{\sigma_1},2(\wp_2+1)p^{\sigma_2}\}$, forcing

$$d_{sp}(C)=\min\{4(\wp_1+1)p^{\sigma_1},2(\wp_2+1)p^{\sigma_2}\}.$$

□

Proposition 3.2.15. *Let* $i < j$ *be such that* $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ *and* $j = p^s$. *Then the symbol-pair distances of* $C = \langle (x^2-\gamma_1)^i (x^2+\gamma_1)^j \rangle$ *are determined as follows:*

$$d_{sp}(C) = 4(\wp_1 + 1)p^{\sigma_1},$$

where $1 \leq \wp_1 \leq p-2$, $0 \leq \sigma_1 \leq s-1$.

Proof. Consider an arbitrary nonzero element $e(x) \in C$. Then there exists a nonzero polynomial $\phi(x) \in R_\lambda$ such that $e(x) = \phi(x)(x^2-\gamma_1)^i(x^2+\gamma_1)^j$. Suppose that t is the largest positive integer with $(x^2-\gamma_1)^t | \phi(x)$ and $\phi(x) = \psi(x)(x^2-\gamma_1)^t$. Then considering the cases, $j + t < p^s$ and $j + t \geq p^s$ and using same technique as Proposition 3.2.14, we have $d_{sp} \geq 4(\wp_1 + 1)p^{\sigma_1}$.

Now, if $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ and $j = p^s$, then $(x^2-\gamma_1)^i(x^2+\gamma_1)^{p^s} \in C$, and we have

$$\begin{aligned}
\mathrm{wt}_{sp}((x^2-\gamma_1)^i(x^2+\gamma_1)^{p^s}) &= \mathrm{wt}_{sp}((x^2-\gamma_1)^i(x^{2p^s}+\gamma_1^{p^s})) \\
&= \mathrm{wt}_{sp}(x^{2p^s}(x^2-\gamma_1)^i) \\
&+ \mathrm{wt}_{sp}(\gamma_1^{p^s}(x^2-\gamma_1)^i) \\
&= 2\,\mathrm{wt}_{sp}((x^2-\gamma_1)^i).
\end{aligned}$$

Using Proposition 3.2.13, we have

$$\mathrm{wt}_{sp}((x^2-\gamma_1)^i(x^2+\gamma_1)^{p^s}) \geq 4(\wp_1 + 1)p^{\sigma_1}.$$

So, we have, $d_{sp}(C) \leq 4(\wp_1+1)p^{\sigma_1}$, forcing $d_{sp}(C) = 4(\wp_1+1)p^{\sigma_1}$. □

Now, we summarize all the above results to give $d_{sp}(C)$ as follows:

Theorem 3.2.16. *Let* $C = \langle (x^2-\gamma_1)^i(x^2+\gamma_1)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s}-\lambda)\rangle}$ *be a* λ*-constacyclic code as in Case* 3 *of Table* 1 *for* $0 \leq i \leq j \leq p^s$*, then the symbol-pair distances* $d_{sp}(C)$ *are completely computed as*

- $d_{sp}(C) = 2$*, if* $i = j = 0$
- $d_{sp}(C) = 4$*, if* $i = 0$ *and* $0 < j \leq p^s$
- $d_{sp}(C) = \min\{2(\wp_1+1)p^{\sigma_1}, 4(\wp_2+1)p^{\sigma_2}\}$*, if* $p^s - p^{s-\sigma_1} + (\wp_1-1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ *and* $p^s - p^{s-\sigma_2} + (\wp_2-1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s - p^{s-\sigma_2} + \wp_2 p^{s-\sigma_2-1}$
- $d_{sp}(C) = 4(\wp_1+1)p^{\sigma_1}$*, if* $p^s - p^{s-\sigma_1} + (\wp_1-1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ *and* $j = p^s$
- $d_{sp}(C) = 0$*, if* $i = j = p^s$

where $0 \leq \sigma_1 \leq \sigma_2 \leq s-1$, $1 \leq \wp_1, \wp_2 \leq p-1$.

Remark 3.2.17. The case with $i \geq j$ has the same symbol-pair distances as Theorem 3.2.16 by symmetries.

3.3 MDS Symbol-Pair Codes over $\Re$

Chee et al. obtained the symbol-pair Singleton bound over the finite field $\mathbb{F}_{p^m}$ in [1].

Corollary 3.3.1. *(Symbol-pair Singleton Bound) [1] Consider a symbol-pair code* C *of length* n *over* $\mathbb{F}_{p^m}$ *with symbol-pair distance* $d_{sp}(C)$*. The symbol-pair Singleton bound is given by* $|C| \leq p^{m(n-d_{sp}(C)+2)}$*.*

Definition 3.3.2. Let C be a symbol-pair code of length n over $\mathbb{F}_{p^m}$. Then C is an MDS symbol-pair code if it meets the Singleton bound for symbol-pair codes.

Using results of Theorems 3.2.10, 3.2.16 and Corollary 3.3.1, we can explore all MDS symbol-pair codes for Case 1, Case 2 and Case 3 of Table 1 and find the MDS symbol-pair codes as follows:

Theorem 3.3.3. *Let* $C = \langle (x^4-\gamma)^j \rangle \subseteq R_\lambda$ *be a* λ*-constacyclic code of length* $4p^s$*. Then the only MDS symbol-pair code is the ambient ring* R_λ *itself.*

Proof. We have the following situations:

Case 1: When $C = \langle 1 \rangle$, then $d_{sp}(C) = 2$. For C to be MDS, $|C| = p^{m(4p^s-d_{sp}(C)+2)}$ i.e., $p^{4mp^s} = p^{m(4p^s-2+2)}$, which is true. Thus, the code C is MDS.

Case 2: When $p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \leq j \leq p^s - p^{s-\sigma} + \wp p^{s-\sigma-1}$, then $d_{sp}(C) = 2(\wp+1)p^\sigma$. C is an MDS code if and only if $|C| = p^{m(4p^s-d_{sp}(C)+2)}$ i.e., $p^{m(4p^s-4j)} = p^{m(4p^s-d_{sp}(C)+2)}$ i.e., $4j = d_{sp}(C) - 2$.

Now,

$$\begin{aligned} j &\geq p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1 \\ 4j &\geq 4(p^s - p^{s-\sigma} + (\wp - 1)p^{s-\sigma-1} + 1) \\ &\geq 4p(p^\sigma - 1) + 4\wp \text{ (equality when } \sigma = s-1) \\ &\geq 4((\wp + 1)p^\sigma) - 4 \text{ (equality when } p - 1 = \wp) \\ &> 2(((\wp + 1)p^\sigma) - 2) \\ &= d_{sp}(C) - 2. \end{aligned}$$

Thus, there is no MDS symbol-pair code.

Case 3: When $C = \langle 0 \rangle$, then $d_{sp}(C) = 0$. For C to be MDS we must have, $|C| = p^{m(4p^s - d_{sp}(C)+2)}$ i.e., $1 = p^{m(4p^s+2)}$ i.e., $2p^s + 1 = 0$, which is false for any s and p.

From the above cases we can get the result. □

Theorem 3.3.4. *Let* $C = \langle (x^2 - \gamma_1)^i (x^2 + \gamma_1)^j \rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle (x^{4p^s} - \lambda) \rangle}$ *be a* λ*-constacyclic code of length* $4p^s$ *over* $\mathbb{F}_{p^m}$*. Then* C *is an MDS symbol-pair code if and only if any one of the following holds:*

1. $i = 0,\ j = 0$ *(trivial MDS)*
2. $i = 0,\ j = 1$ *(non-trivial MDS)*
3. $i = p^s - 1,\ j = p^s$ *(non-trivial MDS)*
4. $i = 1,\ j = 0$ *(non-trivial MDS)*
5. $i = p^s,\ j = p^s - 1$ *(non-trivial MDS).*

Proof. We have the following situations:

Case 1: When $C = \langle 1 \rangle$, then $d_{sp}(C) = 2$. For C to be MDS, $|C| = p^{m(4p^s - d_{sp}(C)+2)}$ i.e., $p^{4mp^s} = p^{m(4p^s-2+2)}$, which is true. Thus, the code C is MDS.

Case 2: When $i = 0, 1 \leq j \leq p^s$, then $d_{sp}(C) = 4$. For C to be MDS, $|C| = p^{m(4p^s - d_{sp}(C)+2)}$ i.e., $p^{m(4p^s-2j)} = p^{m(4p^s-4+2)}$, i.e., $2j = 2$, i.e., $j = 1$. Thus, there is an MDS symbol-pair code for $j = 1$.

Case 3: When $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ and $p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1 \leq j \leq p^s - p^{s-\sigma_2} + \wp_2 p^{s-\sigma_2-1}$, then $d_{sp}(C) = \min\{2(\wp_1+1)p^{\sigma_1}, 4(\wp_2+1)p^{\sigma_2}\}$. C is an MDS code if and only if $|C| = p^{m(4p^s - d_{sp}(C)+2)}$ i.e., $p^{m(4p^s-2i-2j)} = p^{m(4p^s-d_{sp}(C)+2)}$ i.e., $2i + 2j = d_{sp}(C) - 2$.

Now,

$$\begin{aligned}
2i + 2j &\geq 2(p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1) + 2(p^s - p^{s-\sigma_2} + (\wp_2 - 1)p^{s-\sigma_2-1} + 1) \\
&= 4(p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1) \text{ (equality when } \sigma_1 = \sigma_2, \wp_1 = \wp_2) \\
&\geq 4p(p^{\sigma_1} - 1) + 4\wp_1 \text{ (equality when } \sigma_1 = s - 1) \\
&\geq 4(\wp_1 + 1)p^{\sigma_1} - 4 \text{ (equality when } p - 1 = \wp_1) \\
&> 2(\wp_1 + 1)p^{\sigma_1} - 2 \\
&\geq \min\{2(\wp_1+1)p^{\sigma_1}, 4(\wp_2+1)p^{\sigma_2}\} - 2 \\
&= d_{sp}(C) - 2.
\end{aligned}$$

Thus, there is no MDS symbol-pair code.

Case 4: When $p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1 \leq i \leq p^s - p^{s-\sigma_1} + \wp_1 p^{s-\sigma_1-1}$ and $j = p^s$, then $d_{sp}(C) = 4(\wp_1+1)p^{\sigma_1}$. C is an MDS code if and only if $|C| = p^{m(4p^s - d_{sp}(C)+2)}$ i.e., $p^{m(4p^s-2i-2p^s)} = p^{m(4p^s-d_{sp}(C)+2)}$ i.e., $2i = d_{sp}(C) - 2p^s - 2$.

Now,

$$\begin{aligned}2i &\geq 2(p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1)\\ &= 2p^s + 2(p^s - p^{s-\sigma_1} + (\wp_1 - 1)p^{s-\sigma_1-1} + 1) - 2p^s\\ &\geq 4pp^{\sigma_2} - 2p + 2\wp_1 - 2p^s \text{ (equality when } \sigma_1 = s-1)\\ &\geq 4(\wp_1 + 1)p^{\sigma_2} - 2(\wp_1 + 1) + 2\wp_2 - 2p^s \text{ (equality when } p-1 = \wp_1)\\ &= 4((\wp_2 + 1)p^{\sigma_2} - 2p^s - 2\\ &= d_{sp}(C) - 2p^s - 2.\end{aligned}$$

Thus, there is an MDS symbol-pair code for $\sigma_1 = s-1$, $p-1 = \wp_1$ i.e., $i = p^s - 1$.

Case 5: When $C = \langle 0 \rangle$, then $d_{sp}(C) = 0$. For C to be MDS we must have, $|C| = p^{m(4p^s - d_{sp}(C)+2)}$ i.e., $1 = p^{m(4p^s+2)}$ i.e., $2p^s + 1 = 0$, which is false for any s and p.

From the above cases we can get that (1), (2), and (3) are MDS codes for $i \leq j$. By symmetries, (4) and (5) are also MDS codes for $i \geq j$. □

Example 3.3.5. Using MAGMA, we exhibit in Table 2 some examples of non-trivial MDS symbol-pair λ-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$.

Table 2. Some new MDS symbol-pair codes of length $4p^s$ over $\mathbb{F}_{p^m}$

(p, m, s)	λ	$\langle g(x) \rangle$	$[n, k, d_{sp}]$
$(3, 4, 1)$	ω^2	$\langle (x^2 + \omega^{27}) \rangle$	$[12, 10, 4]$
$(3, 2, 2)$	ω^6	$\langle (x^2 - \omega^3)^8 (x^2 + \omega^3)^9 \rangle$	$[36, 2, 36]$
$(3, 4, 1)$	ω^2	$\langle (x^2 - \omega^{27})^3 (x^2 + \omega^{27})^2 \rangle$	$[12, 2, 12]$
$(5, 2, 1)$	3	$\langle (x^2 - \omega^9)^4 (x^2 + \omega^9)^5 \rangle$	$[20, 2, 20]$
$(5, 2, 2)$	ω^{14}	$\langle (x^2 - \omega^7)^{25} (x^2 + \omega^7)^{24} \rangle$	$[100, 2, 100]$
$(5, 3, 1)$	ω^{122}	$\langle (x^2 - \omega^{99})^4 (x^2 + \omega^{99})^5 \rangle$	$[20, 2, 20]$
$(5, 3, 3)$	4	$\langle (x^2 - 2) \rangle$	$[500, 498, 4]$
$(7, 2, 1)$	ω^6	$\langle (x^2 - \omega^{21})^7 (x^2 + \omega^{21})^6 \rangle$	$[28, 2, 28]$
$(11, 2, 2)$	ω^2	$\langle (x^2 - \omega^{11}) \rangle$	$[484, 482, 4]$
$(11, 4, 3)$	ω^{50}	$\langle (x^2 - \omega^{275})^{1331} (x^2 + \omega^{275})^{1330} \rangle$	$[5324, 2, 5324]$
$(13, 2, 2)$	ω^2	$\langle (x^2 - \omega^{91}) \rangle$	$[676, 674, 4]$
$(13, 2, 2)$	4	$\langle (x^2 - 2)^{169} (x^2 + 2)^{168} \rangle$	$[676, 2, 676]$
$(13, 3, 3)$	ω^{506}	$\langle (x^2 + \omega^{461}) \rangle$	$[8788, 8786, 4]$
$(13, 1, 4)$	12	$\langle (x^2 - 5)^{28561} (x^2 + 5)^{28560} \rangle$	$[114244, 2, 114244]$
$(17, 1, 1)$	15	$\langle (x^2 - 7)^{16} (x^2 + 7)^{17} \rangle$	$[68, 2, 68]$
$(29, 1, 1)$	28	$\langle (x^2 - 17) \rangle$	$[116, 114, 4]$
$(29, 1, 1)$	28	$\langle (x^2 - 17)^{28} (x^2 + 17)^{29} \rangle$	$[116, 2, 116]$
$(29, 3, 2)$	ω^{14}	$\langle (x^2 - \omega^{203})^{841} (x^2 + \omega^{203})^{840} \rangle$	$[3364, 2, 3364]$
$(61, 1, 1)$	60	$\langle (x^2 + 11) \rangle$	$[244, 242, 4]$
$(101, 1, 1)$	100	$\langle (x^2 - 10)^{100} (x^2 + 10)^{101} \rangle$	$[404, 2, 404]$

3.4 Conclusion

The symbol distances of several classes of repeated-root constacyclic codes over the finite field $\mathbb{F}_{p^m}$ are determined by using the idea of the coefficient weight of a polynomial. Using distance distributions, we explored some novel MDS code classes as an application. The symbol-pair distances are fully obtained for some classes of λ-constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$. We investigate two new classes of non-trivial MDS symbol-pair codes of length $4p^s$ and demonstrate a number of novel symbol-pair codes of length $4p^s$.

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