Chapter 1

Introduction to Multi-Symbol Constacyclic Codes

Transmission of data across a noisy channel, as well as retrieval of information at the receiving end of the transmission, falls under the purview of coding theory. Coding theory gives straightforward answers to the problems of detecting and correcting data transmission errors across noisy channels. As a result of Shannon's remarkable work [55, 56], the field of coding theory has made significant progress. Richard Hamming was a pioneer in the development and use of error-correcting codes [41]. The relevance of algebra, geometry, etc. in coding theory is widely recognised, with several profound mathematical conclusions being utilised to create coding theory. Algebraic coding theory is a sub-field of discrete applied mathematics related to the development of error-control codes and encoding/decoding techniques. Due to their practical applications, algebraic coding theory is a significant component of mathematics. When it comes to improving the reliability of communication over noisy channels, error-correcting codes have a wide range of applications, including

ISBN: 978-81-989164-0-2
Thesis DOI: 10.21467/thesis.183

the transmission of images from space, the quality of sound on CDs, the reliability of computer networks and wireless communication as well as the identification of monographs and other items by their ISBNs, among other things. Coding theorists are interested in algebraic codes because they are simple to create, encode, and decode. Algebraic codes have traditionally been explored primarily in the context of finite fields. It's worth noting that coding theorists have long been fascinated by codes involving infinite rings. Since Hammons et al. [42] demonstrated that certain non-linear binary codes with excellent parameters are really the binary images under the Gray map of some linear codes over $\mathbb{Z}_4$, the ring of integers modulo 4, codes over finite rings have received considerable attention. With the use of a Gray map, researchers Calderbank et al. [10], Hammons et al. [42], and Nechaev [49] were able to relate many binary non-linear codes to linear codes over the ring $\mathbb{Z}_4$ of integers modulo 4 in the 1990s. In order to better understand linear codes over $\mathbb{Z}_4$ in particular, and linear codes over finite commutative chain rings in general, numerous academics studied linear codes over $\mathbb{Z}_4$.

In the theory of error-correcting codes, constacyclic codes play an important role. Constacyclic codes, moreover, have various real-world applications. Shift registers can effectively encode and decode these codes because they have rich algebraic structures. They also offer excellent error-correction capabilities. All of this clarifies their chosen engineering role. Berlekamp [5] is credited with the introduction and study of constacyclic codes over finite fields. There are several algebraic structures in these codes, which are extensions of cyclic and negacyclic codes in nature. Linear shift registers are a good choice for encoding and decoding these codes because of their efficiency. Many coding theorists were motivated by this specific line of research to further investigate constacyclic codes over finite commutative chain rings, which are an important class of linear codes that have been studied extensively. It has been discovered that, despite all attempts, the algebraic structure of constacyclic codes is only known for a small number of specific lengths and a small number of specific classes of finite commutative chain rings. From [39], it is well known that Λ-constacyclic codes over a finite commutative chain ring $\mathcal{R}$ are categorized as the ideals $\langle h(x)\rangle$ in $\dfrac{\mathcal{R}[x]}{\langle x^n-\Lambda\rangle}$, where Λ is a unit in $\mathcal{R}$ and $h(x)$ divides $x^n-\Lambda$. In the early 1970s, almost all the research was concentrated on simple root codes. In simple root codes, the length n of the code is coprime to the characteristic of the field otherwise, codes are called repeated-root codes. Berman[6] changed the tradition and studied the algebraic structure of repeated-root codes, which were later investigated further in the 1970s and 1980s by researchers such as Massey et al. [48], Falkner et al. [40], Roth and Seroussi [53], among others. Castagnoli et al. [14] and van Lint [45] conducted the most extensive research on repeated-root codes, which resulted in the most generalisation in the 1990s. They demonstrated that repeated-root cyclic codes are asymptotically bad since they have a concatenated structure and have repeated roots. Researchers have been driven to investigate this family of codes because they are optimal in some situations [3, 18, 50, 61, 64]. After that, many researchers came forward and studied such type of codes. Numerous authors [1, 7, 8, 51, 54] have investigated particular classes of repeated-root constacyclic codes over certain classes of finite chain rings since 2003. Recently, we have looked at the description of different classes of constacyclic codes, including cyclic and negacyclic codes, over various types of finite rings, which we have found to be quite interesting. The family of finite rings of the type $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$ has been extensively used as alphabets in study of a variety of constacyclic codes. For instance, the structure of $\mathbb{F}_2+u\mathbb{F}_2$ is intriguing since it is additively equivalent to $\mathbb{F}_4$ and multiplicatively analogous to $\mathbb{Z}_4$. Numerous researchers have investigated it [2, 4, 9, 43, 59, 62]. Readers can also refer to [14, 16, 21, 22, 24, 25, 28, 29, 33, 37, 47, 61] for more information. Dinh defined the algebraic structure in terms of polynomial generators of all constacyclic codes over $\mathbb{F}_{p^m}$ of length $p^s, 2p^s, 3p^s$, and $4p^s$ in a series of papers [22, 23, 26, 28]. These findings have since been expanded to include a wider range of code lengths. Even though Dinh gave algebraic structures of all constacyclic codes of p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ in [27]. In [17], Chen et al. determined the structures of all constacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. In [38], Dinh et al. categorised the structures of all constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$.

One of coding theory's key goals is to develop codes that are simple to encode and decode, capable of detecting and correcting a high number of errors, and include a sufficiently large number of codewords. In other words, the objective is to find codes with efficient encoding and decoding techniques and the greatest possible value of distance for a given code length, code size, and code alphabet cardinality. Numerous distances like Hamming distances, symbol-pair distances, etc. have been established and investigated in coding theory to analyse a code's error-detecting and error-correcting capabilities with respect to various communication channels. Due to applicability to orthogonal modulated channels, the Hamming distance is the most studied distance in coding theory. Singleton [60] established the following upper bound known as Singleton bound on the size M of any block code in terms of the Hamming distance viz. $M \leq q^{n-d_H+1}$, where q is the cardinality of the code alphabet, n denotes the block length, and d_H represents the code's Hamming distance. When a code C of length n over $\mathcal{R}$ meets the above Singleton bound then C is known as a maximum distance separable (MDS) code with respect to the Hamming distance. In the field of error-correction codes, it is generally known that, for each particular code length and dimension, the maximum distance separable (MDS) codes have the largest Hamming distance, i.e., the highest feasible error-correction capabilities. As a result, creating MDS codes is always a popular topic in the field of coding theory. The study of MDS codes having the largest distance is interesting due to their best possible error-correcting capability.

With the advancement of recent high-density data storage techniques, the reading process of data may be lower than that of the process used to store the data. In response to this situation, Cassuto and Blaum [11, 12] devised a novel coding method in which the outputs of the reading process are pairs of successive symbols. Symbol pairing is a kind of coding method that can deal with the situation of errors in the data reading process. Such channels are known as symbol-pair read channels, and the distance associated with them is known as the symbol-pair distance. These channels are better suited for high-density data storage systems where the reader's spatial resolution is insufficient to distinguish between adjacent symbols. If the codeword $\theta = (\theta_0, \theta_1, \ldots, \theta_{n-1})$ is to be transmitted, then the information is read as $((\theta_0, \theta_1), (\theta_1, \theta_2), \ldots, (\theta_{n-1}, \theta_0))$ in symbol-pair read channel. Cassuto and Litsyn [13] created cyclic symbol-pair codes using algebraic techniques, which they published in 2011. Cassuto and Litsyn demonstrated that the symbol-pair distance of a cyclic code with a dimension larger than one and a Hamming distance of d_H is at least $d_H + 2$ by using the Discrete Fourier Transform technique. They also developed symbol-pair codes from cyclic codes and demonstrated that there are symbols-pair codes with rates strictly larger than those of cyclic codes in the Hamming metric with the same relative distance that exists. They demonstrated that a code can repair up to e symbol-pair errors if and only if the distance between the symbol-pairs is at least $2e + 1$. Additionally, they developed methods for deriving symbol-pair codes from Hamming-distance codes and deduced certain constraints on symbol-pair code parameters. They demonstrated how to decode symbol-pair codes created by interleaving the codewords of two Hamming-distance codes. Apart from that, they demonstrated asymptotically that symbol-pair codes exist with rates strictly greater than the best-known Hamming-distance codes. In the field of error-correction codes, it is generally known that, for each particular code length and dimension, the maximum distance separable (MDS) codes have the largest symbol-pair distance, i.e., the highest feasible error-correction capabilities. As a result, creating MDS codes is always a popular topic in the field of coding theory. MDS symbol-pair codes, as an extension of classical MDS codes, provide the best potential error-correction capabilities over the symbol-pair read channel. Chee et al. [15] later established a Singleton-type bound for symbol-pair codes as $M \leq q^{n-d_{sp}+2}$, where q is the cardinality of the code alphabet, n denotes the block length, and d_{sp} represents the code's symbol-pair distance and produced a large number of MDS symbol-pair codes, i.e., codes that meet the Singleton-type bound for the symbol-pair distance. Several researchers have worked on constructing MDS symbol-pair codes after establishing the Singleton bound [15, 19, 44]. After that, Yaakobi et al. [63] generalized symbol-pair channels to b-symbol read channels, where the read operation is performed as a consecutive sequence of b symbols. The corresponding metric is called the b-symbol distance. Ding et al. obtained the b-symbol Singleton bound over the finite field $\mathbb{F}_{p^m}$ as $M \leq q^{n-d_b+b}$, where q is the cardinality of the code alphabet, n denotes the block length, and d_b represents the code's b-symbol distance in [20].

However, only a few studies have been conducted on how to calculate the Hamming, symbol-pair, and b-symbol distances of certain classes of linear codes since, in general, it is difficult to establish the distances. In [34], Dinh obtained the Hamming distances of all the cyclic codes of prime power lengths over $\mathbb{F}_{p^m}$. In [52], the authors then use the finding of [14] to determine the Hamming distances of cyclic codes of length $2p^s$. In [46], the Hamming distances of cyclic codes of length $3p^s$ were calculated using the results of [14] for the case $\gcd(3, p^m - 1) = 1$. Later, in [26, 34], Dinh et al. computed the Hamming distances of all constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$. Dinh et al. computed the symbol-pair distances of all constacyclic codes of length p^s over $\mathbb{F}_{p^m}$ in [31]. The symbol-pair distances of all cyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$ and of all constacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$ were recently obtained by Dinh et al. in [30] and [35] respectively. Later, in [36], Dinh et al. provided the b-symbol distances of a class of repeated-root costacyclic codes for length ηp^s over $\mathbb{F}_{p^m}$. In [32], Dinh et al. calculated the Hamming and the symbol-pair distances of all repeated-root constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. Sharma and Sidana determined the Hamming, symbol-pair and b -symbol distances of all repeated-root constacyclic codes of length p^s over finite commutative chain rings in [57] and [58] respectively. Motivated by all those work, we determine the multi-symbol distances of various classes of repeated-root constacyclic codes over some classes of finite commutative chain rings, in this monograph. Also, we explore various classes of non-trivial MDS codes using these multi-symbol distance distributions. The monograph is organized in the following manner.

An introduction is provided in Chapter 1, followed by some definitions and preliminary findings in Chapter 2. For various classes of constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m}$, the symbol-pair distances are fully determined in Chapter 3. There are new classes of non-trivial MDS symbol-pair codes of length $4p^s$ that we investigate, and we provide a few examples of these codes.

For all constacyclic codes of length $2p^s$ over $\Re = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ with $u^2 = 0$, the symbol-pair distances are completely obtained in Chapter 4. Also, we identify all

MDS symbol-pair constacyclic codes of length $2p^s$ over $\Re$. As examples, several new symbol-pair codes and MDS symbol-pair codes are constructed.

In Chapter 5, the Hamming and b-symbol distances of all constacyclic codes of length $4p^s$ over $\Re$ are thoroughly established for any non-square unit of $\Re$. Codes with new parameters are constructed as examples. We also identified all MDS constacyclic codes of length $4p^s$ among them with respect to the Hamming distance as well as the b-symbol distance. For $b = 4$, we found several nontrivial MDS b-symbol constacyclic codes of Type 3 of length $4p^s$ in the context of b-symbol distances over $\Re$.

The overview and conclusions of the monograph are presented in Chapter 6.

References

[1] T. Abualrub and R. Oehmke, "On the generators of $\mathbb{Z}_4$ cyclic codes of length 2^e," *IEEE Trans. Inform. Theory*, vol. 49, no. 9, pp. 2126–2126, 2003.

[2] M. M. Al-Ashker, "Simplex codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$," *Arabian Journal for Science and Engineering*, vol. 30, no. 2, 2005.

[3] G. K. Bakshi and M. Raka, "A class of constacyclic codes over a finite field," *Finite Fields Appl.*, vol. 18, no. 2, pp. 362–377, 2012.

[4] E. Bannai, M. Harada, T. Ibukiyama, A. Munemasa, and M. Oura, "Type II codes over $\mathbb{F}_2 + u\mathbb{F}_2$ and applications to Hermitian modular forms," in *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, Springer, vol. 73, 2003, pp. 13–42.

[5] E. Berlekamp, "Algebraic coding theory," *McGraw-Hill, New York*, vol. 8, 1968.

[6] S. Berman, "Semisimple cyclic and abelian codes II," *Cybernetics*, vol. 3, no. 3, pp. 17–23, 1967.

[7] T. Blackford, “Negacyclic codes over $\mathbb{Z}_4$ of even length,” *IEEE Trans. Inform. Theory*, vol. 49, no. 6, pp. 1417–1424, 2003.

[8] T. Blackford, “Cyclic codes over $\mathbb{Z}_4$ of oddly even length,” *Discrete Applied Mathematics*, vol. 128, no. 1, pp. 27–46, 2003.

[9] A. Bonnecaze and P. Udaya, “Cyclic codes and self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2$,” *IEEE Trans. Inform. Theory*, vol. 45, no. 4, pp. 1250–1255, 1999.

[10] A. R. Calderbank, A. R. Hammons, P. V. Kumar, N. J. Sloane, and P. Solé, “A linear construction for certain kerdock and preparata codes,” *Bulletin of the American Mathematical Society*, vol. 29, no. 2, pp. 218–222, 1993.

[11] Y. Cassuto and M. Blaum, “Codes for symbol-pair read channels,” *IEEE Trans. Inform. Theory*, vol. 57, pp. 988–992, 2010.

[12] Y. Cassuto and M. Blaum, “Codes for symbol-pair read channels,” *Proceedings of IEEE International Symposium on Information Theory*, vol. 57, no. 12, pp. 8011–8020, 2011.

[13] Y. Cassuto and S. Litsyn, “Symbol-pair codes: Algebraic constructions and asymptotic bounds,” in *2011 IEEE International Symposium on Information Theory Proceedings*, IEEE, 2011, pp. 2348–2352.

[14] G. Castagnoli, J. L. Massey, P. A. Schoeller, and N. von Seemann, “On repeated-root cyclic codes,” *IEEE Trans. Inform. Theory*, vol. 37, no. 2, pp. 337–342, 1991.

[15] Y. M. Chee, L. Ji, H. M. Kiah, C. Wang, and J. Yin, “Maximum distance separable codes for symbol-pair read channels,” *IEEE Trans. Inform. Theory*, vol. 59, no. 11, pp. 7259–7267, 2013.

[16] B. Chen, H. Q. Dinh, and H. Liu, “Repeated-root constacyclic codes of length lp^s and their duals,” *Discrete Appl. Math.*, vol. 177, pp. 60–70, 2014.

[17] B. Chen, H. Q. Dinh, H. Liu, and L. Wang, “Constacyclic codes of length $2p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,” *Finite Fields Appl.*, vol. 37, pp. 108–130, 2016.

[18] B. Chen, Y. Fan, L. Lin, and H. Liu, “Constacyclic codes over finite fields,” *Finite Fields Appl.*, vol. 18, no. 6, pp. 1217–1231, 2012.

[19] B. Chen, L. Lin, and H. Liu, “Constacyclic symbol-pair codes: Lower bounds and optimal constructions,” *IEEE Trans. Inform. Theory*, vol. 63, no. 12, pp. 7661–7666, 2017.

[20] B. Ding, T. Zhang, and G. Ge, “Maximum distance separable codes for b-symbol read channels,” *Finite Fields Appl.*, vol. 49, pp. 180–197, 2018.

[21] H. Q. Dinh, “Constacyclic codes of length 2^s over Galois extension rings of $\mathbb{F}_2 + u\mathbb{F}_2$,” *IEEE Trans. Inform. Theory*, vol. 55, no. 4, pp. 1730–1740, 2009.

[22] H. Q. Dinh, “Repeated-root constacyclic codes of length $2p^s$,” *Finite Fields Appl.*, vol. 18, no. 1, pp. 133–143, 2012.

[23] H. Q. Dinh, “On repeated-root constacyclic codes of length $4p^s$,” *Asian-Eur. J. Math.*, vol. 6, no. 02, p. 1 350 020, 2013.

[24] H. Q. Dinh, “Complete distances of all negacyclic codes of length 2^s over $\mathbb{Z}^a$,” *IEEE Trans. Inform. Theory*, vol. 53, no. 1, pp. 147–161, 2006.

[25] H. Q. Dinh, “Repeated-root constacyclic codes of length 2^s over $\mathbb{Z}_{2^a}$,” *Contemporary Mathematics*, vol. 419, p. 95, 2006.

[26] H. Q. Dinh, “On the linear ordering of some classes of negacyclic and cyclic codes and their distance distributions,” *Finite Fields Appl.*, vol. 14, no. 1, pp. 22–40, 2008.

[27] H. Q. Dinh, “Constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,” *Journal of Algebra*, vol. 324, no. 5, pp. 940–950, 2010.

[28] H. Q. Dinh, “Structure of repeated-root constacyclic codes of length $3p^s$ and their duals,” *Discrete Mathematics*, vol. 313, no. 9, pp. 983–991, 2013.

[29] H. Q. Dinh, “Repeated-root cyclic and negacyclic codes of length $6p^s$,” *Ring Theory Appl*, vol. 609, p. 69, 2014.

[30] H. Q. Dinh, B. T. Nguyen, and S. Sriboonchitta, “MDS symbol-pair cyclic codes of length $2p^s$ over $\mathbb{F}_{p^m}$,” *IEEE Trans. Inform. Theory*, vol. 66, no. 1, pp. 240–262, 2019.

[31] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and S. Sriboonchitta, “On the symbol-pair distance of repeated-root constacyclic codes of prime power lengths,” *IEEE Trans. Inform. Theory*, vol. 64, no. 4, pp. 2417–2430, 2017.

[32] H. Q. Dinh, B. T. Nguyen, A. K. Singh, and S. Sriboonchitta, “Hamming and symbol-pair distances of repeated-root constacyclic codes of prime power lengths over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,” *IEEE Communications Letters*, vol. 22, no. 12, pp. 2400–2403, 2018.

[33] H. Q. Dinh and H. D. Nguyen, “On some classes of constacyclic codes over polynomial residue rings.,” *Advances in Mathematics of Communications*, vol. 6, no. 2, pp. 175–191, 2012.

[34] H. Q. Dinh, X. Wang, H. Liu, and S. Sriboonchitta, “On the Hamming distances of repeated-root constacyclic codes of length $4p^s$,” *Discrete Mathematics*, vol. 342, no. 5, pp. 1456–1470, 2019.

[35] H. Q. Dinh, X. Wang, H. Liu, and S. Sriboonchitta, “On the symbol-pair distances of repeated-root constacyclic codes of length $2p^s$,” *Discrete Mathematics*, vol. 342, no. 11, pp. 3062–3078, 2019.

[36] H. Q. Dinh, X. Wang, and J. Sirisrisakulchai, “Optimal b-symbol constacyclic codes with respect to the Singleton bound,” *Journal of Algebra and Its Applications*, vol. 19, no. 08, p. 2 050 151, 2020.

[37] H. Q. Dinh and S. R. López-Permouth, “Cyclic and negacyclic codes over finite chain rings,” *IEEE Trans. Inform. Theory*, vol. 50, no. 8, pp. 1728–1744, 2004.

[38] H. Dinh, S Dhompongsa, and S Sriboonchitta, “On constacyclic codes of length $4p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$,” *Discrete Math.*, vol. 340, pp. 832–849, 2017.

[39] S. T. Dougherty, *Algebraic coding theory over finite commutative rings.* Springer, 2017.

[40] G Falkner, B Kowol, W Heise, and E Zehendner, “On the existence of cyclic optimal codes,” *Atti Sem. Mat. Fis. Univ. Modena*, vol. 28, pp. 326–341, 1979.

[41] R. W. Hamming, “Error detecting and error correcting codes,” *Bell System Technical Journal*, vol. 29, no. 2, pp. 147–160, 1950.

[42] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. Sloane, and P. Solé, “The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals, and related codes,” *IEEE Trans. Inform. Theory*, vol. 40, no. 2, pp. 301–319, 1994.

[43] W. C. Huffman, “On the decomposition of self-dual codes over $\mathbb{F}_2+u\mathbb{F}_2$ with an automorphism of odd prime order,” *Finite Fields Appl.*, vol. 13, no. 3, pp. 681–712, 2007.

[44] X. Kai, S. Zhu, and P. Li, “A construction of new MDS symbol-pair codes,” *IEEE Trans. Inform. Theory*, vol. 61, no. 11, pp. 5828–5834, 2015.

[45] J. H. van Lint, “Repeated-root cyclic codes,” *IEEE Trans. Inform. Theory*, vol. 37, no. 2, pp. 343–345, 1991.

[46] L. Liu, L. Li, X. Kai, and S. Zhu, “Repeated-root constacyclic codes of length $3lp^s$ and their dual codes,” *Finite Fields Appl.*, vol. 42, pp. 269–295, 2016.

[47] S. R. López-Permouth, H. Özadam, F. Özbudak, and S. Szabo, “Polycyclic codes over Galois rings with applications to repeated-root constacyclic codes,” *Finite Fields Appl.*, vol. 19, no. 1, pp. 16–38, 2013.

[48] J. L. Massey, D. J. Costello, and J. Justesen, “Polynomial weights and code constructions,” *IEEE Trans. Inform. Theory*, vol. 19, no. 1, pp. 101–110, 1973.

[49] A. A. Nechaev, “Kerdock code in a cyclic form,” *Discrete Mathematics and Applications*, 1991.

[50] C.-S. Nedeloaia, “Weight distributions of cyclic self-dual codes,” *IEEE Trans. Inform. Theory*, vol. 49, no. 6, pp. 1582–1591, 2003.

[51] G. H. Norton and A. Sălăgean, “On the structure of linear and cyclic codes over a finite chain ring,” *Applicable algebra in engineering, communication and computing*, vol. 10, no. 6, pp. 489–506, 2000.

[52] H. Özadam and F. Özbudak, “The minimum Hamming distance of cyclic codes of length $2p^s$,” in *International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes*, Springer, 2009, pp. 92–100.

[53] R Roth and G. Seroussi, “On cyclic MDS codes of length q over $GF(q)$(corresp.),” *IEEE Trans. Inform. Theory*, vol. 32, no. 2, pp. 284–285, 1986.

[54] A. Sălăgean, “Repeated-root cyclic and negacyclic codes over a finite chain ring,” *Discrete Appl. Math.*, vol. 154, no. 2, pp. 413–419, 2006.

[55] C. E. Shannon, "A Mathematical Theory of Communication," *Bell System Technical Journal*, vol. 27, no. 3, pp. 379–423, 1948.

[56] C. E. Shannon, "A Mathematical Theory of Communication," *Bell System Technical Journal*, vol. 27, no. 4, pp. 623–656, 1948.

[57] A. Sharma and T. Sidana, "On the structure and distances of repeated-root constacyclic codes of prime power lengths over finite commutative chain rings," *IEEE Trans. Inform. Theory*, vol. 65, no. 2, pp. 1072–1084, 2018.

[58] A. Sharma and T. Sidana, "On b-symbol distances of repeated-root constacyclic codes," *IEEE Trans. Inform. Theory*, vol. 65, no. 12, pp. 7848–7867, 2019.

[59] I. Siap, "Linear codes over $\mathbb{F}_2 + u\mathbb{F}_2$ and their complete weight enumerators," *Codes and Designs, Columbus, OH*, vol. 10, pp. 259–271, 2000.

[60] R. Singleton, "Maximum distance q-nary codes," *IEEE Trans. Inform. Theory*, vol. 10, no. 2, pp. 116–118, 1964.

[61] L.-Z. Tang, C. B. Soh, and E. Gunawan, "A note on the q-ary image of a q^m-ary repeated-root cyclic code," *IEEE Trans. Inform. Theory*, vol. 43, no. 2, pp. 732–737, 1997.

[62] P. Udaya and A. Bonnecaze, "Decoding of cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$," *IEEE Trans. Inform. Theory*, vol. 45, no. 6, pp. 2148–2157, 2006.

[63] E. Yaakobi, J. Bruck, and P. H. Siegel, "Constructions and decoding of cyclic codes over b-symbol read channels," *IEEE Trans. Inform. Theory*, vol. 62, no. 4, pp. 1541–1551, 2016.

[64] K.-H. Zimmermann, "On generalizations of repeated-root cyclic codes," *IEEE Trans. Inform. Theory*, vol. 42, no. 2, pp. 641–649, 1996.