Chapter 8:

Gamma Rayleigh Distribution: Properties and Application

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Additional information is available at the end of the chapter

Introduction

In recent years generating new distributions to analyse different life time data has received considerable attention. A number of methods are available in literature that can be used to generalise the existing models to make them more flexible. Among various diverse generalizing methods available, the generalization of our interest is T-X family of distribution by Alzetreeh *et al* (2012). . Let $r(t)$ be the PDF of a non-negative continuous random variable T defined on $[0, \infty)$, and let $F(x)$ denote the CDF of a random variable X. Then the CDF for the T-X family of distributions for random variable X is

$$G(x) = \int_0^{-\log(1-F(x))} r(t)\,dt \quad . \tag{8.1}$$

And the corresponding PDF is given by

$$g(x) = \frac{f(x)}{1-F(x)} r\{-\log(1-F(x))\} \quad . \tag{8.2}$$

Let T follow gamma distribution with PDF

$$r(t) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-\frac{t}{\beta}} t^{\alpha-1} \quad ;\alpha, \beta > 0, t \geq 0 \ . \tag{8.3}$$

Using (8.3) in (8.1) we obtain the CDF of Gamma-X family of distribution given as

$$G(x) = \frac{\gamma(\alpha, -\log(1-F(x))/\beta)}{\Gamma(\alpha)} \quad . \tag{8.4}$$

Where $\gamma(\alpha, x) = \int_0^x z^{\alpha-1} e^{-z} dz$ is the incomplete gamma function.

The corresponding PDF of Gamma-X family is given by

$$G(x)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\, f(x)\{-\log(1-F(x))\}^{\alpha-1}(1-F(x))^{\frac{1}{\beta}-1} \tag{8.5}$$

A number of distributions have been developed using gamma-X generalization. A few among them are: Gamma-Pareto Distribution and Its Applications by Alzaatreh *et al.* (2014), the gamma-normal distribution: Properties and applications by Alzaatreh *et al.* (2012) etc.

In this context we propose an extension of Rayleigh distribution known as *Gamma- Rayleigh distribution* (GRD for short) using gamma-X family of distribution in order to make the distribution more flexible to real life data. The outline of this paper is as follows: in section 8.2, the PDF and CDF of proposed distribution i.e., GRD is derived. Various statistical properties of the distribution such as moments, moment generating function, mode etc. are discussed in section 8.3. The reliability measures of the distribution are discussed in section 8.4. The expressions for different information measures of the distribution is obtained in section 8.5. In section 8.6, expressions for mean deviation and median are derived. The parameter estimation of the parameters of the distribution is discussed in section 8.7. In section 8.8 the application of the proposed model is debated using real life examples and finally some conclusions and discussions are given at the end.

Derivation of GRD

The cumulative distribution function (CDF) of Rayleigh distribution is given by

$$F(x)=1-e^{-\frac{x^2}{2\theta^2}} \tag{8.6}$$

The probability density function (PDF) of Rayleigh distribution is given by

$$f(x)=\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}\ ;\ \theta>0, x>0 \tag{8.7}$$

Using (8.6) in (8.4) we get, the CDF of GRD given by

$$G(x)=\frac{\gamma\left(\alpha,\frac{x^2}{2\theta^2\beta}\right)}{\Gamma(\alpha)} \tag{8.8}$$

and the corresponding PDF of GRD is given by

$$g(x)=\frac{x}{\theta^2\Gamma(\alpha)\beta^{\alpha}}e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1};\alpha,\beta,\theta>0, x>0 \tag{8.9}$$

If we put $\alpha=1$ and $\beta=1$ in (8.9) we get the PDF of Rayleigh distribution.

Statistical Properties and Reliability Measures

In this section, the basic statistical properties of the proposed distribution are investigated.

Moments

The r^{th} moment about origin can be obtained as

$$\mu_r' = \int_0^\infty x^r \, g(x)\, dx$$

$$= \int_0^\infty x^r \frac{x}{\theta^2 \Gamma(\alpha)\beta^\alpha} e^{-x^2/2\beta\theta^2} \left(\frac{x^2}{2\theta^2}\right)^{\alpha-1} dx$$

$$= \frac{(2\theta^2\beta)^{r/2}\,\Gamma\left(\alpha+\frac{r}{2}\right)}{\Gamma(\alpha)} \tag{8.10}$$

Where $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz$ is the gamma function.

Putting r=1, 2, 3, 4 in (8.10) we can obtain first four moments about origin.

Mean and Variance of GRD

The mean and variance of the GRD is given as

Mean $= \dfrac{(2\theta^2\beta)^{1/2}\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}$ and variance $= \dfrac{(2\theta^2\beta)}{\Gamma(\alpha)}\left[\Gamma(\alpha+1) - \dfrac{\left(\Gamma\left(\alpha+\frac{1}{2}\right)\right)^2}{\Gamma(\alpha)}\right]$

Moment Generating function

The moment generating function of the GRD can be derived as

$$M_X(t) = E\left(e^{tx}\right)$$

$$M_X(t) = \int_0^\infty e^{tx} g(x) dx$$

$$M_X(t) \;=\int_0^\infty (1+tx+\frac{(tx)^2}{2!}+\cdots)\,f(x)dx$$

$$M_X(t) \;=\sum_{r=0}^\infty \frac{t^r}{r!}\int_0^\infty x^r g(x)dx$$

$$M_X(t) \;=\sum_{r=0}^\infty \frac{t^r}{r!}\mu_r{}'$$

Using (10) in above equation, we get

$$M_X(t)=\sum_{r=0}^\infty \frac{t^r}{r!}\frac{(2\theta^2\beta)^{r/2}\Gamma\left(\alpha+\frac{r}{2}\right)}{\Gamma(\alpha)}$$

Mode

The mode of the GRD can be obtained as

$$\frac{\partial}{\partial x}\log g(x)=0$$

$$\Rightarrow \frac{\partial}{\partial x}\log\left[\frac{x}{\theta^2\Gamma(\alpha)\beta^\alpha}e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1}\right]=0$$

$$\Rightarrow x=\theta(\beta(2\alpha-1))^{\frac{1}{2}}$$

Harmonic mean

The harmonic mean of the GRD can be obtained as

$$H..M \;=\int_0^\infty \frac{1}{x}g(x)\,dx$$

$$=\int_0^\infty \frac{1}{x}\frac{x}{\theta^2\Gamma(\alpha)\beta^\alpha}e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1}dx$$

$$=\frac{\Gamma\left(\alpha-\frac{1}{2}\right)}{(2\theta^2\beta)^{1/2}\Gamma(\alpha)}\;;\alpha>\frac{1}{2}$$

Reliability analysis of GRD

Survival and Failure Rate Functions

The survival function, hazard rate function and reverse hazard rate function associated with the GRD is given by Eqn. (8.11), (8.12) and (8.13) respectively

$$S(x)=\frac{\Gamma\left(\alpha, {x^2}/{2\theta^2\beta}\right)}{\Gamma(\alpha)} \tag{8.11}$$

$$h(x)=\frac{x}{\theta^2\beta^\alpha\Gamma\left(\alpha,\frac{x^2}{2\theta^2\beta}\right)}e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1} \tag{8.12}$$

$$\tau(x)=\frac{x}{\theta^2\beta^\alpha\gamma\left(\alpha,\frac{x^2}{2\theta^2\beta}\right)}e^{-x^2/2\beta\theta^2}\left(\frac{x^2}{2\theta^2}\right)^{\alpha-1} \tag{8.13}$$

Mean Residual Time and Mean Waiting Time

The mean residual time is given by

$$\mu(t)=E(T-t\,|\,T>0)=\frac{1}{S(t)}\left(E(t)-\int_0^t xg(x)dx\right)-t$$

The mean residual time of GRD is given as

$$\mu(t)=(2\theta^2\beta)^{1/2}\frac{\Gamma\left(\alpha+\frac{1}{2},\frac{t^2}{2\beta\theta^2}\right)}{\Gamma\left(\alpha,\frac{t^2}{2\beta\theta^2}\right)}-t$$

Also, the mean waiting time which is the waiting time elapsed since the failure of an item given that that this failure has happened in the interval [0, t] is given by

$$\bar{\mu}(t)=t-\frac{1}{F(t)}\int_0^t xg(x)dx$$

The mean waiting time of GRD is given as

$$\bar{\mu}(t) = t - \frac{(2\theta^2\beta)^{\frac{1}{2}}\gamma\left(\alpha + \frac{1}{2}, \frac{t^2}{2\beta\theta^2}\right)}{\gamma\left(\alpha, \frac{t^2}{2\beta\theta^2}\right)}$$

The graphs of PDF, CDF and hazard function for different value of the parameter are given below

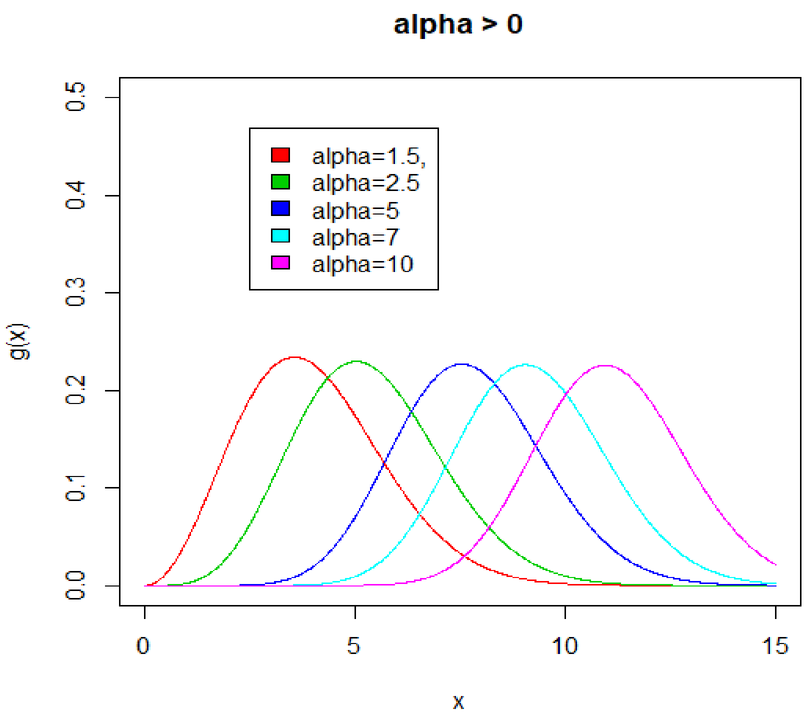


Figure 8.1: *Graph of density function*

Fig 8.1 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter α when β *and* θ are fixed. Fig 8.2 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter β when α *and* θ are fixed. Fig 8.3 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter θ when β *and* α are fixed. Fig 8.4 represents Graphs of Probability density function of Gamma Rayleigh distribution for different values of parameter α, β *and* θ. Fig 8.5 represents Graphs of hazard function of Gamma Rayleigh distribution for different values of parameter α, β *and* θ.

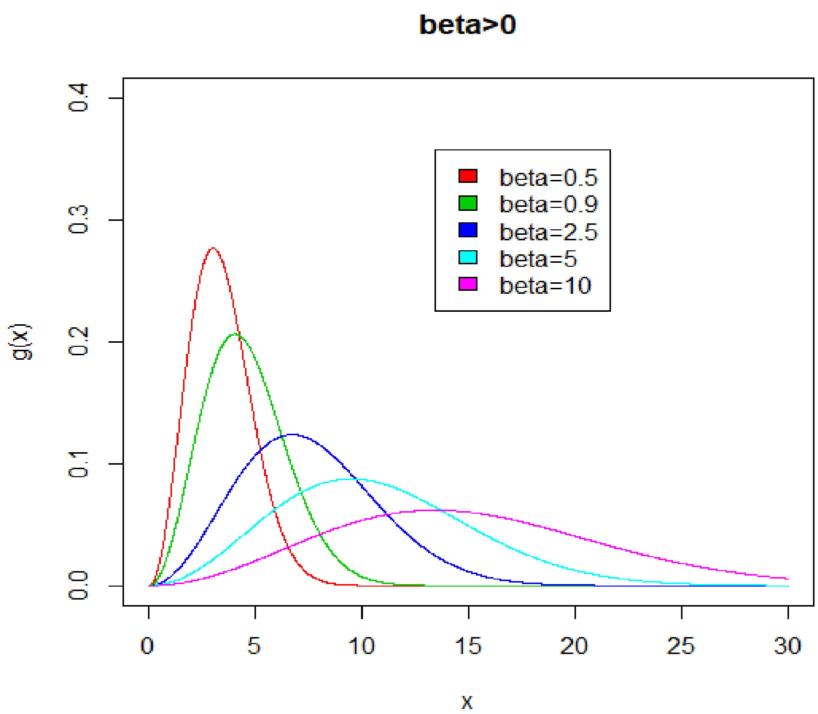


Figure 8.2: *Graph of density function*

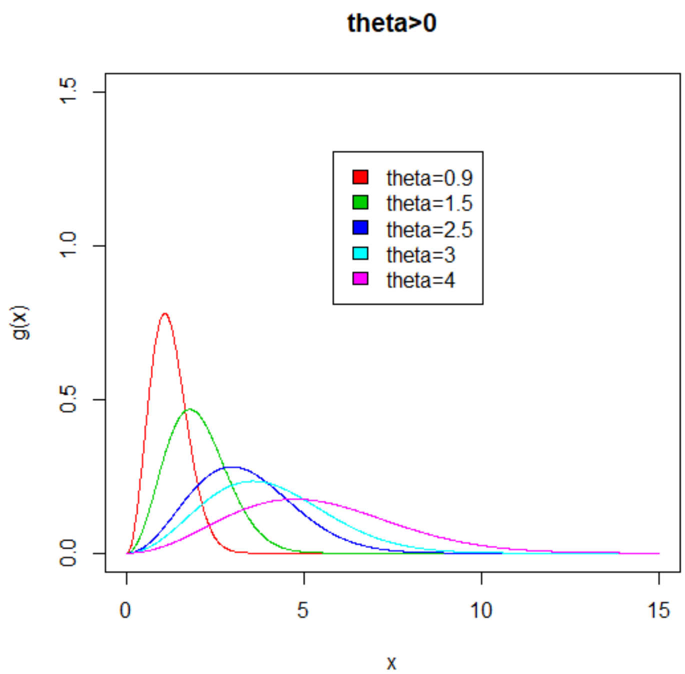


Figure 8.3: *Graph of density function*

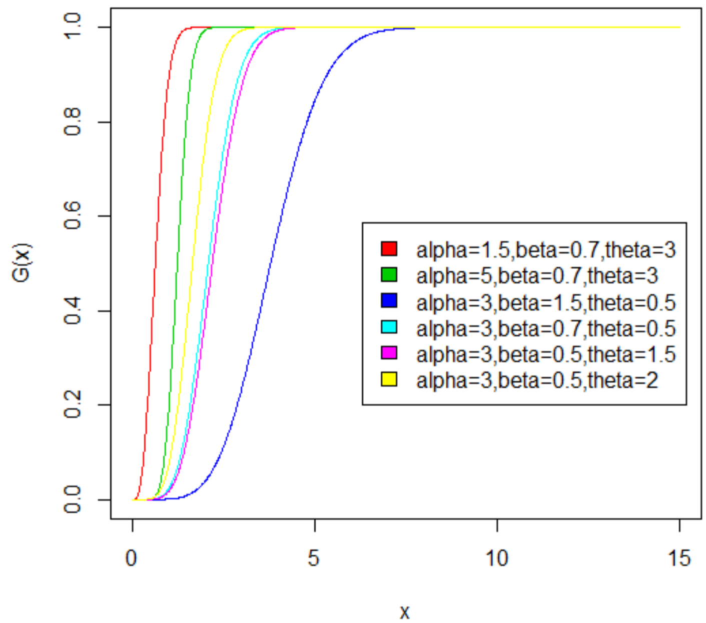


Figure 8.4: *Graph of density function*

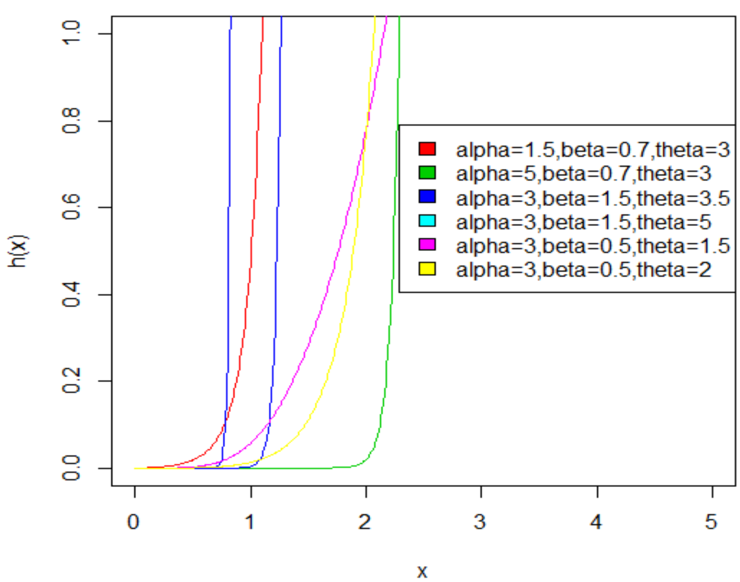


Figure 8.5: *Graph of density function*

Information Measures

Renyi Entropy

The Renyi entropy is denoted by $I_R(\rho)$ and is defined as:

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty} f(x)^{\rho}\, dx\right\};\ \rho > 0, \rho \neq 1$$

Therefore, the Renyi entropy for GRD is given as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\left(\frac{2}{\theta^2\beta}\right)^{\frac{(\rho-1)}{2}} \frac{\Gamma\left(\rho\left(\alpha-\frac{1}{2}\right)+\frac{1}{2}\right)}{(\Gamma(\alpha))^{\rho}\,\rho^{\left(\rho\left(\alpha-\frac{1}{2}\right)+\frac{1}{2}\right)}}\right\};\ \alpha > \frac{1}{2}$$

Shannon Entropy

The Shannon entropy is defined as

$$\eta_X = E[-\log f(x)]$$

The Shannon entropy for gamma –x family is given as

$$\eta_X = -E\left[\log f(F^{-1}(1-e^{-T}))\right] + \alpha(1-\beta) + \log\beta + \log\Gamma(\alpha) + (1-\alpha)\psi(\alpha) \qquad (8.14)$$

Where ψ the digamma function and T the gamma random variable with parameters α and β (see Alzaatreh, *et al.* [3] for proof details).

We have

$$-E(\log f(F^{-1}(1-e^{T}))) = -\frac{1}{2} - \frac{1}{2}(\psi(\alpha) + \log\beta) + \log\theta + \alpha\beta \qquad (8.15)$$

Using (8.15) in (8.14) we get the Shannon entropy of GRD as given below

$$\eta_X = \frac{1}{2}\log\beta + \log\theta + \alpha + \log\Gamma(\alpha) - \frac{1}{2} + (\frac{1}{2} - \alpha)\psi(\alpha)$$

Mean Deviation About Mean and Median

The mean deviation about mean of the GRD can be obtained as

$$D(\mu) = \int_0^{\mu}(\mu - x)g(x)\,dx + \int_{\mu}^{\infty}(x-\mu)g(x)\,dx$$

$$D(\mu) = 2\int_0^{\mu} (\mu - x) g(x)\, dx$$

$$D(\mu) = \frac{2\mu\gamma\left(\alpha, \frac{\mu^2}{2\beta\theta^2}\right)}{\Gamma(\alpha)} - \frac{2^{3/2}(\theta^2\beta)^{1/2}\gamma\left(\alpha + \frac{1}{2}, \frac{\mu^2}{2\beta\theta^2}\right)}{\Gamma(\alpha)}$$

The mean deviation about median of the GRD can be obtained as

$$D(M) = \int_0^{M} (M - x) g(x)\, dx + \int_M^{\infty} (x - M) g(x)\, dx$$

$$D(M) = \mu - 2\int_0^{M} x\, g(x)\, dx$$

$$D(M) = \mu - \frac{2^{3/2}(\theta^2\beta)^{1/2}\gamma\left(\alpha + \frac{1}{2}, \frac{M^2}{2\beta\theta^2}\right)}{\Gamma(\alpha)}$$

Parameter Estimation

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from the GRD then the log likelihood function is given by

$$\log L = \sum_{i=1}^{n} \log x_i - 2n\log\theta - n\log\Gamma(\alpha) - n\alpha\log\beta - \sum_{i=1}^{n}\left(\frac{x_i^2}{2\theta^2\beta}\right) + (\alpha - 1)\sum_{i=1}^{n}\log\left(\frac{x_i^2}{2\theta^2}\right)$$

(8.16)

Taking the derivative of the natural logarithm of the likelihood function (8.16) w.r.t α, β and θ respectively and equation to zero we get the following three equations:

$$\frac{\partial}{\partial\alpha}\log L = -n\psi(\alpha) - n\log\beta + \sum_{i=1}^{n}\log\left(\frac{x_i^2}{2\theta^2}\right) = 0 \quad (8.17)$$

Where $\psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$

$$\frac{\partial}{\partial\beta}\log L = -\frac{n\alpha}{\beta} + \sum_{i=1}^{n}\left(\frac{x_i^2}{2\theta^2\beta^2}\right) = 0 \quad (8.18)$$

$$\frac{\partial}{\partial\theta}\log L = -\frac{2n\alpha}{\theta} + \sum_{i=1}^{n}\left(\frac{x_i^2}{\theta^3\beta}\right) = 0 \quad (8.19)$$

The MLE's of the parameters α, β and θ can be obtained by solving system of equations (8.17), (8.18) and (8.19). Methods such as Newton –Raphson technique can be used to solve these non-linear equations.

Application

In this section, three real life data sets are used to demonstrate the usefulness of GRD. The analysis is performed by using R Software. The distribution that are being used for comparison purpose with the proposed model are

1. Rayleigh distribution (RD) given by Rayleigh (1980) with PDF

$$g(x,\theta)=\frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}} \quad ;\theta>0,x>0$$

2. Weibull- Rayleigh distribution (WRD) given by Ahmad *et al.* (2017). with PDF

$$g(x,\alpha,\beta,\theta)=\frac{\alpha x^2}{\beta\theta^2}\left(\frac{x^2}{2\beta\theta^2}\right)^{\alpha-1} e^{-\left(\frac{x^2}{2\beta\theta^2}\right)^{\alpha}} \;;\alpha,\beta,\theta>0,x>0$$

Data set 8.1: This data set were used by Birnbaum and Saunders (1969) and correspond to the fatigue time of 101 6061-T6 aluminium coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps).

Table 8.1: *MLE's estimates, AIC, BIC, AICC for the fitted models to the Data set*

Distribution	MLE			-Log L	AIC	BIC	AICC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$				
GRD	11.652	8.459	9.7927	423.75	853.51	861.21	853.71
RD			97.2239	504.21	1010.4	1012.9	1010.6
WRD	3.2083	20.602	22.575	433.88	873.77	881.46	873.97

Data set 8.2: The data are the exceedances of flood peaks (in m3/s) of the Wheaton River near Car cross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place. This data were analysed by Choulakian and Stephens (2001) and are given below

1.7, 2.2 ,14.4 ,1.1 ,0.4, 20.6, 5.3 ,0.7, 13.0, 12.0, 9.3 ,1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1 ,2.5, 14.4, 1.7 37.6 0.6 2.2, 39.0 ,0.3 ,15.0 ,11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5 ,2.5, 27.0, 1.9, 2.8.

Table 8.2: *MLE's estimates, AIC, BIC, AICC for the fitted models to the Data set*

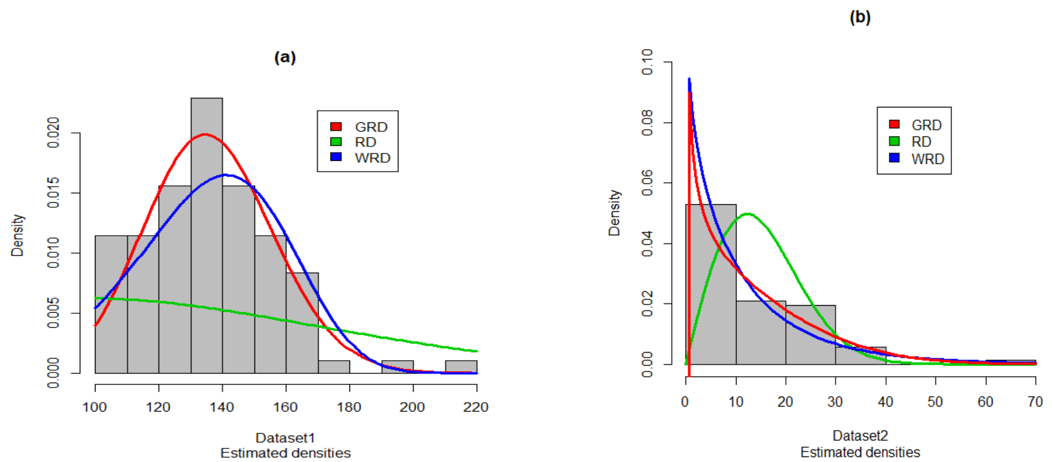
Distribution	MLE			-Log L	AIC	BIC	AICC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$				
GRD	0.324	6.408	8.4712	251.276	508.553	515.383	508.75
RD			12.207	607.675	609.952	609.952	607.87
WRD	0.450	4.091	4.066	251.498	508.997	515.827	509.19

Data set 8.3: The data set represents the lifetime data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975).

1.1,1.4,1.3,1.7,1.9,1.8,1.6,2.2,1.7,2.7,4.1,1.8,1.5,1.2,1.4,3,1.7,2.3,1.6,2.

Table 8.3: *MLE's estimates, AIC, BIC, AICC for the fitted models to the Data set*

Distribution	MLE			-Log L	AIC	BIC	AICC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$				
GRD	2.3428	0.0528	4.0596	19.170	44.340	47.327	44.540
RD			1.4284	22.478	46.957	47.953	47.157
WRD	1.3935	0.2219	3.1969	20.586	47.172	50.160	47.372



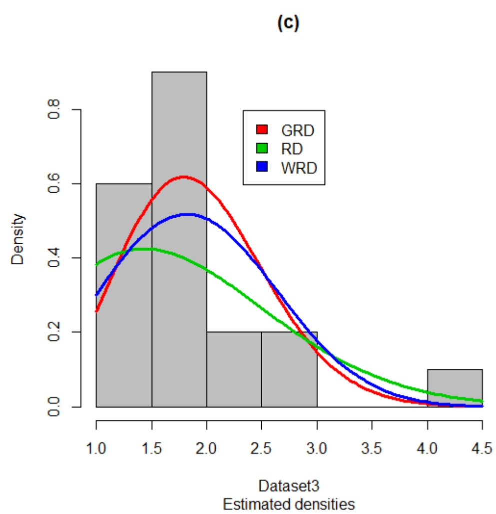


Figure 6: *(a) Plots of the estimated PDF of GRD and other competitive models for Data set 1.*
(b) Plots of the estimated PDF of GRD and other competitive models for Data set 2.

(c) Plots of the estimated PDF of GRD and other competitive models for Data set 3.

Discussion

Since the model with the least value of AIC, AICC, BIC are considered to be best fit, it can be seen from Table 8.1, Table 8.2 and Table 8.3 that GRD has the least value of AIC, AICC and BIC for all the data sets. Hence GRD fits the given data sets quite well as compared other models used for comparison. The histograms of the three data sets and the estimated PDF's of the proposed and competitive models are displayed in Figure 6(a), Figure 6(b) and Figure 6(c).

Conclusion

In this paper we have successfully defined a three parameter Gamma Rayleigh distribution based on T-X family of distribution introduced by Alzetreeh *et al* (2012). Some of the structural properties including moments, mgf, and harmonic mean are studied. Also the entropy estimation of proposed Distribution is carried out. The parameters involved in the distribution are estimated by maximum likelihood method. The flexibility of this model is illustrated by means of three real life data sets and it is evident that the Gamma Rayleigh distribution provides better fit than Inverse Rayleigh distribution.

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