Chapter 5:

Estimating the Parameter of Weighted Ailamujia Distribution using Bayesian Approximation Techniques

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Introduction

Ailamujia distribution (also known as Эрланга distribution) was proposed by Lv et al [1] for applicability in various engineering fields. He studied various descriptive measures of the newly developed lifetime model which include mean, variance, median and maximum likelihood estimate. Pan et al [2] considered this distribution for estimating intervals and testing of hypothesis. Long [3] obtained the Bayes' estimates of Эрланга Distribution under Type II censoring for missing data with three different priors. Li [4] estimated the parameters of Ailamujia model considering the three loss functions under a non informative prior. Uzma et al. [5] proposed the weighted Ailamujia Distribution and applied in reliability analysis. Assume X denotes the life span of a product following the Weighted Ailamujia distribution, its probability density function and cumulative density function is given respectively as follows:

$$f(x,\theta) = \frac{(2\theta)^{c+2}}{\Gamma(c+2)} x^{c+1} e^{-2\theta x}; \qquad x \geq 0, \theta \geq 0. \qquad (5.1)$$

where θ the shape parameter and c are the weight parameter

$$F(x,\theta) = \frac{1}{\Gamma(c+2)} \gamma(c+2, 2\theta x) \ ; \quad x \geq 0, \theta \geq 0. \qquad (5.2)$$

The likelihood function and the corresponding log likelihood of (5.1) are given in the equations (5.3) and (5.4) respectively as:

$$L(x \mid \theta) \propto \theta^{n(c+2)} e^{-2\theta \sum_{i=1}^{n} x_i}. \qquad (5.3)$$

$$\log L(x \mid \theta) = n(c+2)\log \theta - 2\theta \sum_{i=1}^{n} x_i. \tag{5.4}$$

The major focus of the current manuscript is to examine the performance of unknown shape parameter θ of Weighted Ailamujia distribution under a variety of priors using the two Bayesian approximation techniques.

Bayesian Approximation Techniques of Posterior Modes

Bayesian paradigm gives a comprehensive model for updating the prior information in view of the current knowledge. Those who like the elegance of Bayesian outlook study important properties of the posterior and predictive distributions. If the resulting distribution is in closed form and difficult to characterize it, analytical or numerical approximation methods are often used for accuracy with less computational complicacy. Many authors have reviewed the approximation methods including Sultan and Ahmad [6, 7] for Kumaraswamy distribution and generalized Power function distribution, Kawsar and Ahmad [8] for Inverse Exponential and Uzma and Ahmad [9,10] for Inverse Lomax and Dagum distributions.

Normal Approximation

In Bayesian approach, approximation techniques for large samples usually consider the normal approximation to the posterior distribution. If the posterior distribution is less skewed with sharp peak, the most convenient way is to approximate it by normal distribution. This posterior distribution is usually localized near the posterior mode and behaves normal under different conditions when the sample size is increased. The Normal approximation for the posterior distribution $P(\theta \mid x)$ centered at mode is given as

$$P(\theta \mid x) \sim N\left[\hat{\theta}, \left\{I(\hat{\theta})\right\}^{-1}\right] \tag{5.5}$$

where $I(\hat{\theta}) = -\dfrac{\partial^2}{\partial \theta^2} \log P(\theta \mid x)$ \hfill (5.6)

Under Jeffery's Prior $g_1(\theta) \propto \dfrac{1}{\theta}$, the posterior distribution is given by

$$P_1(\theta \mid x) \propto g_1(\theta)L(x \mid \theta) \tag{5.7}$$

$$\Rightarrow \quad P_1(\theta \mid x) \propto \theta^{n(c+2)-1} e^{-2\theta T}, \text{ where } T = \sum_{i=1}^{n} x_i. \tag{5.8}$$

As such $\hat{\theta} = \dfrac{\partial}{\partial \theta} \log P_1(\theta \mid x) = \dfrac{n(c+2)-1}{2T}$ and $\left[I(\hat{\theta})\right]^{-1} = \dfrac{[n(c+2)-1]}{(2T)^2}$.

Therefore, $P_1(\theta \mid x) \sim N\left[\dfrac{[n(c+2)-1]}{2T}, \dfrac{n(c+2)-1}{(2T)^2}\right].$ (5.9)

Similarly, under Gamma Prior $g_2(\theta) \propto e^{-b_1\theta}\theta^{a_1-1}$, the posterior distribution is given by $P_2(\theta \mid x) \propto \theta^{n(c+2)+b_1-1}e^{-\theta[2T+a_1]}$ (8) and can be approximated as

$$P_2(\theta \mid x) \sim N\left[\frac{n(c+2)+a_1-1}{2T+b_1}, \frac{n(c+2)+a_1-1}{(2T+b_1)^2}\right].$$ (5.10)

Also, under the Erlang prior $g_3(\theta) \propto \theta^{a_2}e^{-\theta b_2}$, the posterior distribution is $\theta^{n(c+2)+a_2}e^{-\theta(2T+b_2)}$ (5.10) and can be approximated as

$$P_3(\theta \mid x) \sim N\left[\frac{n(c+2)+a_2}{2T+b_2}, \frac{n(c+2)+a_2}{(2T+b_2)^2}\right].$$ (5.11)

T-K Approximation

Laplace's method proves to be an efficient procedure for solving the difficult integrals which arise in mathematics. For approximating the average values of functions of parameters and marginal densities, Laplace's method is generally used. It is widely applicable method in statistics for its simpler computations than the MCMC methods etc. Moreover, Laplace method provides better view of the problem. The different manuscripts in literature which describe the method include Lindley [11], Tierney and Kadane [12] and Leonard, Huss and Tsui [13]. Tierney and Kadane presented the Laplace method for computing $E[h(\theta) \mid x]$ as

$$E[h(\theta) \mid x] \cong \frac{\hat{\phi}^* \exp\left[-nh^*\left(\hat{\theta}^*\right)\right]}{\phi \exp\left[-nh\left(\hat{\theta}\right)\right]}$$ (5.12)

where $-nh(\hat{\theta}) = \log P(\theta \mid x);\ -nh^*(\hat{\theta}^*) = \log P(\theta \mid x) + \log h(\theta).$

$\hat{\phi}^2 = -\left[-nh''(\hat{\theta})\right]^{-1};\ \phi^{*2} = -\left[-nh''^*(\hat{\theta}^*)\right]^{-1}.$

Thus, for Weighted Ailamujia distribution, T-K approximation for shape parameter θ under different priors is obtained as:

Under Jeffery's Prior $g_1(\theta) \propto \dfrac{1}{\theta}$, the posterior distribution is given by the equation (6)

We have $nh(\theta) = \log P_1(\theta \mid x) = [n(c+2)-1]\log\theta - 2\theta T$.

For which, $nh'(\theta) = \dfrac{n(c+1)-1}{\theta} - 2T \Rightarrow \hat{\theta} = \dfrac{n(c+2)-1}{2T}$.

Also, $nh''(\hat{\theta}) = \dfrac{-[n(c+2)-1]}{\theta^2} = \dfrac{-(2T)^2}{[n(c+2)-1]}$.

$\therefore \hat{\phi}^2 = \dfrac{n(c+2)-1}{(2T)^2} \Rightarrow \hat{\phi} = \dfrac{\sqrt{n(c+2)-1}}{2T}$.

Now, $nh^*(\theta^*) = -nh(\theta) + \log h(\theta)$, where $h(\theta) = \theta$

$\Rightarrow \qquad nh^*(\theta^*) = n(c+2)\log\theta - 2\theta T$.

Then, $nh'^*(\theta^*) = \dfrac{n(c+2)}{\theta} - 2T \Rightarrow \hat{\theta}^* = \dfrac{n(c+2)}{2T}$.

Further, $nh''^*(\hat{\theta}) = \dfrac{-(2T)^2}{n(c+2)} \Rightarrow \hat{\phi}^* = \dfrac{\sqrt{n(c+2)}}{2T}$

$$E[\theta \mid x] = \left(\frac{n(c+2)}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}} \frac{n(c+2)}{2T}\exp(-1).$$

$$E(\theta^2 \mid x) = \frac{\hat{\phi}^{**}\left[nh^{**}(\hat{\theta})\right]}{\hat{\phi}\left[nh(\hat{\theta})\right]}, \text{here } h(\theta) = \theta^2. \tag{5.13}$$

As such, $E[\theta^2 \mid x] = \left(\dfrac{n(c+2)+1}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}} \left[\dfrac{n(c+2)+1}{2T}\right]^2 \exp(-2)$.

$$Var(\theta \mid x) = \left(\frac{n(c+2)+1}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}}\left[\frac{n(c+2)+1}{2T}\right]^2 \exp(-2) - \left\{\left(\frac{n(c+2)}{n(c+2)-1}\right)^{n(c+2)-\frac{1}{2}}\frac{n(c+2)}{2T}\exp(-1)\right\}^2. \tag{5.14}$$

Also, under Gamma Prior, the posterior distribution is given by the equation (5.9) and the estimates are given by

$$E(\theta \mid x) = \left[\frac{n(c+2)+a_1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1}{2T+b_1}\right]\exp(-1). \tag{5.15}$$

$$E(\theta^2 \mid x) = \left[\frac{n(c+2)+a_1+1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1+1}{2T+b_1}\right]^2\exp(-2)$$

$$V(\theta \mid x) = \left[\frac{n(c+2)+a_1+1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1+1}{2T+b_1}\right]^2\exp(-2)-$$

$$\left\{\left[\frac{n(c+2)+a_1}{n(c+2)+a_1-1}\right]^{n(c+2)+a_1-\frac{1}{2}}\left[\frac{n(c+2)+a_1}{2T+b_1}\right]\exp(-1)\right\}^2. \tag{5.16}$$

Similarly, under Erlang Prior $g_4(\theta) \propto \theta^{a_2}e^{-\theta b_2}$, the posterior distribution is given by the equation (5.10) and we have

$$E(\theta \mid x) = \left[\frac{n(c+2)+a_2+1}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+1}{(2T+b_2)}\right]\exp(-1). \tag{5.17}$$

$$E(\theta^2 \mid x) = \left[\frac{n(c+2)+a_2+2}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+2}{2T+b_2}\right]^2\exp(-2).$$

$$Var(\theta \mid x) = \left[\frac{n(c+2)+a_2+2}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+2}{2T+b_2}\right]^2\exp(-2)-$$

$$\left\{\left[\frac{n(c+2)+a_2+1}{n(c+2)+a_2}\right]^{n(c+2)+a_2+\frac{1}{2}}\left[\frac{n(c+2)+a_2+1}{(2T+b_2)}\right]\exp(-1)\right\}^2. \tag{5.18}$$

Applications

For comparing the efficacy of different priors and the two approximation techniques for the weighted Ailamujia distribution, we have considered the three real life data sets related to engineering field.

Data Set 5. 1: The first data set is provided in Murthy et al. [14] about time between failures for 30 repairable items. The data are listed as the following:

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45,0.70,1.06,1.46, 0.30,1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17

DataSet 5.2: The second data represent the strength data measured in GPA, for single carbon fibres and impregnated 1000 carbon fiber tows reported by Badar and Priest [15]. We will be considering the single fibres of 10 mm in gauge length with sample sizes 63.

0.101,0.322,0.403,0.428,0.457,0.550,0.561,0.596,0.597,0.645,0.654,0.674,0.718,0.722,0.725,0.723,0.775,0.814,0.816,0.818,0.824,0.859,0.875,0.938,0.940,1.056,1.117,1.128,1.137,1.137,1.177,1.196,1.230,1.325,1.339,1.345,1.420,1.423,1.435,1.443,1.464,1.472,1.494,1.532,1.546,1.577,1.608,1.635,1.693,1.701,1.737,1.754,1.762,1.828,2.052,2.071,2.086,2.171,2.224,2.227,2.425,2.595,2.220

Data set 5.3: The third data set is on the strengths of 1.5 cm glass fibres. The data was originally obtained by workers at the UK National Physical Laboratory and it has been used by Bourguignon *et al.* [16].

Table 5.1: *Posterior estimates for Normal Approximation*

	c	Jeffery's Prior	Gamma Prior			Erlang Prior		
			$a_1=b_1=0.5$	$a_1=b_1=1.0$	$a_1=b_1=2.0$	$a_2=b_2=0.5$	$a_2=b_2=1.0$	$a_2=b_2=2.0$
Data Set I	0.5	0.79948 (0.00863)	0.80055 (0.00860)	0.80162 (0.00856)	0.80372 (**0.00849**)	0.81130 (0.00871)	0.81231 (0.00868)	0.81429 (0.00861)
	1	0.96153 (0.01038)	0.96174 (0.01033)	0.96194 (0.01028)	0.96235 (**0.01017**)	0.97249 (0.01045)	0.97263 (0.01039)	0.97292 (0.01028)
	2	1.28565 (0.01388)	1.28411 (0.01379)	1.28259 (0.01370)	1.27961 (**0.01353**)	1.29486 (0.01391)	1.29328 (0.01382)	1.29018 (0.01364)

	c	Jeffery's Prior	Gamma $a_1=b_1=0.5$	Gamma $a_1=b_1=1.0$	Gamma $a_1=b_1=2.0$	Erlang $a_2=b_2=0.5$	Erlang $a_2=b_2=1.0$	Erlang $a_2=b_2=2.0$
Data Set II	0.5	0.99914 (0.00637)	0.99914 (0.00635)	0.99914 (0.00633)	0.99915 (**0.00629**)	1.00551 (0.00639)	1.00549 (0.00637)	1.00545 (0.00633)
	1	1.20025 (0.00766)	1.19961 (0.00763)	1.19898 (0.00760)	1.19772 (**0.00755**)	1.20597 (0.00767)	1.20532 (0.00764)	1.20402 (0.00758)
	2	1.60246 (0.01023)	1.60054 (0.01018)	1.59864 (0.01014)	1.59486 (**0.01005**)	1.60690 (0.01022)	1.60498 (0.01018)	1.60117 (0.01009)
Data Set III	0.5	0.82429 (0.00434)	0.82475 (0.00433)	0.82521 (0.00432)	0.82612 (**0.00430**)	0.83000 (0.00436)	0.83045 (0.00435)	0.83133 (0.00433)
	1	0.99020 (0.00521)	0.99022 (0.00520)	0.99025 (0.00518)	0.99030 (**0.00516**)	0.99548 (0.00522)	0.99549 (0.00521)	0.99551 (0.00518)
	2	1.32202 (0.00696)	1.32118 (0.00694)	1.32034 (0.00691)	1.31867 (**0.00687**)	1.32643 (0.00696)	1.32557 (0.00694)	1.32388 (0.00690)

Table 5.2: *Posterior estimates for TK approximation*

	c	Jeffery's Prior	Gamma Prior $a_1=b_1=0.5$	Gamma Prior $a_1=b_1=1.0$	Gamma Prior $a_1=b_1=2.0$	Erlang Prior $a_2=b_2=0.5$	Erlang Prior $a_2=b_2=1.0$	Erlang Prior $a_2=b_2=2.0$
Data Set I	0.5	0.81029 (0.00875)	0.81131 (0.00871)	0.81232 (0.00868)	0.81430 (**0.00861**)	0.81667 (0.01766)	0.81765 (0.01759)	0.81957 (0.01744)
	1	0.97235 (0.01050)	0.97250 (0.01045)	0.97264 (0.01039)	0.97293 (**0.01028**)	0.97785 (0.02113)	0.97797 (0.02102)	0.97821 (0.02080)
	2	1.29646 (0.01400)	1.29487 (0.01391)	1.29329 (0.01382)	1.29019 (**0.01364**)	1.30023 (0.02805)	1.29862 (0.02787)	1.29547 (0.02751)
Data Set II	0.5	1.005532 (0.00641)	1.00551 (0.00639)	1.00549 (0.00637)	1.00546 (**0.00633**)	1.00869 (0.01287)	1.00866 (0.01283)	1.00860 (0.01275)
	1	1.20663 (0.00770)	1.20598 (0.00767)	1.20532 (0.00764)	1.20403 (**0.00758**)	1.20915 (0.01543)	1.20849 (0.01537)	1.20718 (0.01525)
	2	1.60884 (0.01027)	1.60691 (0.01022)	1.60498 (0.01018)	1.60117 (**0.01009**)	1.61009 (0.02053)	1.60815 (0.020443)	1.60432 (0.020266)

Data Set III	0.5	0.82956 (0.00436)	0.83000 (0.00436)	0.83045 (0.00435)	0.83133 (**0.00433**)	0.83263 (0.00877)	0.83307 (0.00875)	0.83394 (0.00872)
	1	0.99547 (0.00524)	0.99548 (0.00522)	0.99549 (0.00521)	0.99551 (**0.00518**)	0.99810 (0.01051)	0.99811 (0.01048)	0.99812 (0.01043)
	2	1.32729 (0.00699)	1.32643 (0.00696)	1.32558 (0.00694)	1.32388 (**0.00690**)	1.32906 (0.01399)	1.32819 (0.01394)	1.32648 (0.01385)

Conclusion

From table 5.1 and 5.2, it is clearly evident that the posterior variance of gamma prior is less than the other priors under both the approximation techniques especially when the value of both the hyper parameters a_1 and b_1 is taken as 2. Further, it is also noted that the values of normal approximation are less than the T-K approximation for all the three data sets.

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