Chapter 4:

Bayesian Inference of Ailamujia Distribution Using Different Loss Functions

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Introduction

Ailmujia distribution is proposed by Lv et al. (2002). Pan et al. (2009) studied the interval estimation and hypothesis test of Ailamujia distribution based on small sample. Uzma et al. (2017) studied the weighted version Ailamujia distribution. The cumulative distribution function of Ailamujia distribution is given by

$$F(x;\theta,\alpha)=1-(1+2\theta x)e^{-2\theta x} \quad ,x\geq 0,\theta>0 \tag{4.1}$$

and the probability density function (pdf) corresponding to (4.1) is

$$f(x;\theta,\alpha)=4x\theta^2 e^{-2\theta x} \quad ,x\geq 0\ ,\theta>0 \tag{4.2}$$

Our objective in this study is to find the Bayes estimators of the parameter of Ailamujia distribution using non-informative Jeffery's prior and informative Gamma prior under squared error loss function, Entropy loss function and LINEX loss function. Finally, an application is considered to equate the performance of these estimates under different loss functions by manipulative posteriors risk using R Software.

Material and Methods

Recently Bayesian estimation technique has established great contemplation by most researchers. Bayesian analysis is a significant approach to statistics, which properly seeks use of prior information and Bayes Theorem provides the formal basis for using this information.

In this paper we consider the Jeffrey's prior proposed by Al-Kutubi (2005) as:

$$g(\theta) \propto \sqrt{I(\theta)} \tag{4.3}$$

where $[I(\theta)] = -nE\left[\frac{\partial^2 \log f(x;\theta)}{\partial \theta^2}\right]$ is the Fisher's information matrix. For the model (4.2),

$g(\theta) = k\frac{1}{\theta}$, where k is a constant.

The second prior which we have used is gamma prior i.e

$$g(\theta) \propto \frac{\alpha^\beta}{\Gamma\beta} e^{-\alpha\theta} \theta^{\beta-1} \tag{4.4}$$

with the above priors, we use three different loss functions for the model (4.2), viz squared error loss function which is symmetric, and Entropy and LINEX loss function which are asymmetric loss functions.

Maximum Likelihood Estimation

Let x_1, x_2,…, x_n be a random sample of size n from Ailamujia distribution, then the log likelihood function can be written as

$$\log L(\theta,\lambda) = n\log 4 + 2n\log\theta + \sum_{i=1}^{n} x_i - 2\theta\sum_{i=1}^{n} x_i \tag{4.5}$$

the ML estimator of θ is obtained by solving the equation

$$\frac{\partial \log L(\theta)}{\partial \theta} = 0$$

$$\Rightarrow \frac{2n}{\theta} - 2\sum_{i=1}^{n} x_i = 0 \qquad \Rightarrow \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i}$$

Bayesian estimation of Ailamujia distribution under Assumption of Jeffrey's prior

Consider n recorded values, $x = (x_1, x_2, ..., x_n)$ having probability density function as

$$f(x;\theta,\alpha) = 4x\theta^2 e^{-2\theta x}$$

we consider the prior distribution of θ to be Jeffrey's prior i.e $g(\theta) \propto \frac{1}{\theta}$

The posterior distribution of θ under the assumption of Jeffrey's prior is given by

$$\pi(\theta/x) \propto L(x/\theta)\, g(\theta)$$

$$\Rightarrow \pi(\theta/x) \propto (4\theta)^n \prod_{i=1}^{n} x_i\, e^{-2\theta\sum_{i=1}^{n} x_i} \frac{1}{\theta}$$

$$\Rightarrow \pi(\theta/x) = k\,\theta^{2n-1}\, e^{-2\theta\sum_{i=1}^{n} x_i}$$

where k is independent of θ

and $$k^{-1} = \int_0^{\infty} \theta^{2n-1}\, e^{-2\theta\sum_{i=1}^{n} x_i}\, d\theta \qquad \Rightarrow k^{-1} = \frac{\Gamma 2n}{\left[2\sum_{i=1}^{n} x_i\right]^{2n}}$$

Hence posterior distribution of θ is given by

$$\pi(\theta/x) = \frac{\left[2\sum_{i=1}^{n} x_i\right]^{2n}}{\Gamma 2n}\, \theta^{2n-1}\, e^{-2\theta\sum_{i=1}^{n} x_i}$$

$$\pi(\theta/x) = \frac{t^{2n}}{\Gamma 2n}\, \theta^{2n-1}\, e^{-t\theta} \tag{4.6}$$

where $t = 2\sum_{i=1}^{n} x_i$

Estimator Under Squared Error Loss Function

By using squared error loss function $l(\hat{\theta},\theta) = c_1(\hat{\theta}-\theta)^2$ for some constant c_1 the risk function is given by

$$R(\hat{\theta},\theta) = E\left[\, l(\hat{\theta},\theta)\right]$$

$$= \int_0^{\infty} c_1(\hat{\theta}-\theta)^2\, \frac{t^{2n}}{\Gamma 2n}\, \theta^{2n-1}\, e^{-t\theta}\, d\theta$$

$$=\frac{c_1 t^{2n}}{\Gamma 2n}\left[\hat{\theta}^2\frac{\Gamma 2n}{t^{2n}}+\frac{\Gamma(2n+2)}{t^{2n+2}}-2\hat{\theta}\frac{\Gamma(2n+1)}{t^{2n+1}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}}=0$, we obtain the Baye's estimator as

$$\hat{\theta}_{Js}=\frac{n}{t}, \quad \text{where } t=2\sum_{i=1}^{n}x_i \tag{4.7}$$

Estimator Under Entropy Loss Function

Using entropy loss function $L(\delta)=a[\delta-\log(\delta)-1]\,;a>0,\ \delta=\frac{\hat{\theta}}{\theta}$, the risk function is given by

$$R(\hat{\theta},\theta)=\int_0^\infty a[\delta-\log(\delta)-1]\ \frac{(t)^{2n}}{\Gamma(2n)}\theta^{2n-1}e^{-t\theta}\ d\theta$$

$$=\frac{a t^{2n}}{\Gamma(2n)}\left[\hat{\theta}\frac{\Gamma(2n-1)}{(t)^{2n-1}}-\log\hat{\theta}\frac{\Gamma(2n)}{(t)^{2n}}+\frac{\Gamma'(2n)}{t^{2n}}-\frac{\Gamma 2n}{t^{2n}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}}=0$, we obtain the Baye's estimator as

$$\hat{\theta}_{JE}=\frac{2n-1}{t}, \quad \text{where } t=2\sum_{i=1}^{n}x_i \tag{4.8}$$

Estimator Under LINEX Loss Function

Using LINEX loss function $l(\theta,\hat{\theta})=\exp\left\{b\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1$ for some constant *b* the risk function is given by

$$R(\hat{\theta},\theta)=\int_0^\infty\left(\exp\left\{b_1\left(\hat{\theta}-\theta\right)\right\}-b_1\left(\hat{\theta}-\theta\right)-1\right)\frac{t^{2n}}{\Gamma(2n)}\theta^{2n-1}\ e^{-t\theta}\ d\theta$$

$$=\frac{t^{2n}}{\Gamma(2n)}\left[e^{b\hat{\theta}}\frac{\Gamma(2n)}{(b+t)^{2n}}-b\hat{\theta}\frac{\Gamma(2n)}{t^{2n}}+b\frac{\Gamma(2n+1)}{t^{2n+1}}-\frac{\Gamma(2n)}{t^{2n}}\right]$$

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Now solving $\dfrac{\partial R(\hat{\theta},\theta)}{\partial \hat{\theta}} = 0$, we obtain the Bayes estimator as

$$\hat{\theta}_{JL} = \frac{1}{b}\log\left(\frac{b+t}{t}\right)^{2n} \tag{4.9}$$

Bayesian Estimation of Ailamujia Distribution Under Assumption of Gamma Prior

Consider n recorded values, $x = (x_1, x_2, ..., x_n)$ having probability density function as

$$f(x;\theta,\alpha) = 4x\theta^2 e^{-2\theta x}$$

we consider the prior distribution of θ to be Gamma prior i.e $g(\theta) \propto \dfrac{\alpha^\beta}{\Gamma\beta} e^{-\alpha\theta}\theta^{\beta-1}$

The posterior distribution of θ under the assumption of Gamma prior is given by

$$\pi(\theta/x) \propto L(x/\theta)\, g(\theta)$$

$$\Rightarrow \pi(\theta/x) \propto (4\theta)^n \prod_{i=1}^{n} x_i\, e^{-2\theta\sum_{i=1}^{n} x_i} \frac{\alpha^\beta}{\Gamma\beta} e^{-\alpha\theta}\theta^{\beta-1}$$

$$\Rightarrow \pi(\theta/x) = k\,\theta^{2n+\beta-1} e^{-\left(\alpha+2\sum_{i=1}^{n} x_i\right)\theta}$$

where k is independent of θ

and $k^{-1} = \int_0^\infty \theta^{2n+\beta-1} e^{-\left(\alpha+2\sum_{i=1}^{n} x_i\right)\theta} d\theta \quad \Rightarrow k^{-1} = \dfrac{\Gamma(2n+\beta)}{\left(\alpha+2\sum_{i=1}^{n} x_i\right)^{2n+\beta}}$

Henceforth posterior distribution of θ is given by

$$\pi(\theta/x) = \frac{\left(\alpha+2\sum_{i=1}^{n} x_i\right)^{2n+\beta}}{\Gamma(2n+\beta)} \theta^{2n+\beta-1} e^{-\left(\alpha+2\sum_{i=1}^{n} x_i\right)\theta}$$

$$\pi(\theta/x)=\frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\theta^{2n+\beta-1}\,e^{-(\alpha+t)\theta} \tag{4.10}$$

where $t=2\sum_{i=1}^{n}x_i$

Estimator Under Squared Error Loss Function

By using squared error loss function $l(\hat{\theta},\theta)=c_1(\hat{\theta}-\theta)^2$ for some constant c_1 the risk function is given by

$$R(\hat{\theta},\theta)=E\left[\,l(\hat{\theta},\theta)\right]$$

$$=\int_0^\infty c_1(\hat{\theta}-\theta)^2\,\frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\theta^{2n+\beta-1}\,e^{-(\alpha+t)\theta}\;d\theta$$

$$=\frac{c_1(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\left[\hat{\theta}^2\int_0^\infty\theta^{2n+\beta-1}\,e^{-(\alpha+t)\theta}\,d\theta+\int_0^\infty\theta^{2n+\beta+1}\,e^{-(\alpha+t)\theta}\,d\theta-2\hat{\theta}\int_0^\infty\theta^{2n+\beta}\,e^{-(\alpha+t)\theta}\,d\theta\right]$$

$$=\frac{c_1(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\left[\hat{\theta}^2\frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}}+\frac{\Gamma(2n+\beta+2)}{(\alpha+t)^{2n+\beta+2}}-2\hat{\theta}\frac{\Gamma(2n+\beta+1)}{(\alpha+t)^{2n+\beta+1}}\right]$$

Now solving $\dfrac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Bayes estimator as

$$\hat{\theta}_{Gs}=\frac{2n+\beta}{\alpha+t},\quad\text{where } t=2\sum_{i=1}^{n}x_i \tag{4.11}$$

Estimator Under Entropy Loss Function

Using entropy loss function $L(\delta)=a[\delta-\log(\delta)-1]\,;a>0,\ \delta=\dfrac{\hat{\theta}}{\theta}$, the risk function is given by

$$R(\hat{\theta},\theta)=\int_0^\infty a[\delta-\log(\delta)-1]\;\frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\theta^{2n+\beta-1}\,e^{-(\alpha+t)\theta}\;d\theta$$

$$=\frac{a(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\left[\hat{\theta}\frac{\Gamma(2n+\beta-1)}{(\alpha+t)^{2n+\beta-1}}-\log\hat{\theta}\frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}}+\frac{\Gamma'(2n+\beta)}{(\alpha+t)^{2n+\beta}}-\frac{\Gamma 2n+\beta}{(\alpha+t)^{2n+\beta}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Baye's estimator as

$$\hat{\theta}_{GE}=\frac{2n+\beta-1}{\alpha+t}, \quad \text{where } t=2\sum_{i=1}^{n}x_i \tag{4.12}$$

Estimator Under LINEX Loss Function

Using LINEX loss function $l(\theta,\hat{\theta})=\exp\{b(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1$ for some constant b the risk function is given by

$$R(\hat{\theta},\theta)=\int_0^\infty\left(\exp\{b(\hat{\theta}-\theta)\}-b_1(\hat{\theta}-\theta)-1\right)\frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\theta^{2n+\beta-1}\; e^{-(\alpha+t)\theta}\; d\theta$$

$$=\frac{(\alpha+t)^{2n+\beta}}{\Gamma(2n+\beta)}\left[e^{b\hat{\theta}}\frac{\Gamma(2n+\beta)}{(\alpha+b+t)^{2n+\beta}}-b\hat{\theta}\frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}}+b\frac{\Gamma(2n+\beta+1)}{(\alpha+t)^{2n+\beta+1}}-\frac{\Gamma(2n+\beta)}{(\alpha+t)^{2n+\beta}}\right]$$

Now solving $\frac{\partial R(\hat{\theta},\theta)}{\partial\hat{\theta}}=0$, we obtain the Bayes estimator as

$$\hat{\theta}_{GL}=\frac{1}{b}\log\left(\frac{b+\alpha+t}{\alpha+t}\right)^{2n+\beta} \tag{4.13}$$

Application

The data set was initially stated by Badar Priest (1982) on failure stresses (inGpa) of 65 single carbon fibers of length 50mm respectively. The data set is given as follows: 1.339,1.434,1.549,1.574,1.589,1.613,1.746,1.753,1.764,1.807,1.812,1.84,1.852,1.852,1.862,1.864,1.931,1.952,1.974,2.019,2.051,2.055,2.058,2.088,2.125,2.162,2.171,2.172,2.18,2.194,2.211,2.27,2.272,2.28,2.299,2.308,2.335,2.349,2.356,2.386,2.39,2.41,2.43,2.458,2.471,2.497,2.514,2.558,2.577,2.593,2.601,2.604,2.62,2.633,2.67,2.682,2.699,2.705,2.735,2.785,3.02,3.042,3.116,3.174. This data set had used by Al-Mutairi (2013) and Uzma et al. (2017).

The posterior estimates and posterior risks are calculated, and result is presented in table 4.1 and table 4.2.

Table 4.1: *Posterior estimates and Posterior variances using Jeffery's Prior*

	$\hat{\theta}_S$	$\hat{\theta}_L$		$\hat{\theta}_E$
		b = 0.5	b =1.0	
Posterior Estimates	0.2231	0.0003	0.0007	0.4427
Posterior Risks	0.0513	7.4402	2.1083	5.6629

Table 4.2: *Posterior estimates and Posterior variances using Gamma Prior*

	$\hat{\theta}_S$	$\hat{\theta}_L$		$\hat{\theta}_E$
		b = 0.5	b =1.0	
Posterior Estimates	0.5266	0.05432	0.0574	0.6434
Posterior Risks	0.1430	7.4023	2.0032	5.9721

It is clear from Table 4.1and Table 4.2, on equating the Bayes posterior risk of dissimilar loss functions, it is observed that the squared error loss function has less Bayes posterior risk in both non informative and informative priors than other loss functions. According to the decision rule of less Bayes posterior risk we accomplish that squared error loss function is more preferable loss function.

Conclusion

We have predominantly considered the Bayes estimator of the parameter of Ailamujia distribution using Jeffrey's prior and gamma prior supposing three different loss functions. The Jeffrey's prior gives the prospect of covering wide continuum of priors to get Bayes estimates of the parameter. From the results, we observe that in most cases, Bayesian Estimator under Squared error Loss function has the smallest posterior risk values for both prior's i.e, Jeffrey's and gamma prior information.

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