Chapter 3:

Mathematical Model of Accelerated Life Testing Plan Using Geometric Process

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DOI: https://doi.org/10.21467/books.44.3

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Introduction

In ALT analysis, the life-stress relationship is generally used to estimate the parameters of failure time distribution at use condition which is just a re-parameterization of original parameters but from statistical point of view, it is easy and reasonable to deal with original parameters directly instead of developing inferences for the parameters of the life stress relationship. It can be seen that the original parameter of the distribution can be directly handled by the assumption that the lifetime of the items formd GP at increasing level of stress. The concept of geometric process in accelerated life testing was first introduced by Lam (1988) in repair replacement problem. Since then many authors have studied maintenance problem and system reliability by using GP model. Lam (2007) used geometric model to study a multistate system and inferred a policy for replacement that minimizes the average cost per unit time for long run. After that a lot of works have been done and the available literature shows that the GP model is one of the simplest among the available models for the study of data with a single or multiple trend, e.g., Lam and Zhang (1996), Lam (2005). Zhang (2008) studied repairable system with delayed repair by using the GP repair model. Huang (2011) analyzed the complete and censored data for exponential distribution applying the model of GP. Zhou et al. (2012) extended the GP model for Rayleigh distribution for the progressive type I hybrid censored data in ALT. Kamal et al. (2013) analyzed the complete samples for Pareto distribution with constant stress accelerated life testing plan by using geometric process model. Anwar et al. (2013) used the process to analyze the model of ALT for Marshal-Olkin extended exponential distribution, then extended her work for type I censored data (Anwar et al., 2014).

This chapter deals with constant stress accelerated life testing for generalized exponential distribution using geometric process with complete data. Estimates of parameters are obtained by maximum likelihood estimation technique and confidence intervals for parameters are obtained by using the asymptotic properties. Lastly, statistical properties of estimates and confidence intervals are examined through a simulation study.

The Model

The Geometric Process

A stochastic process $\{X_n, n = 1,2,...\}$ is said to be a geometric process if there is a real valued $\lambda(>0)$ in such a way that $\{\lambda^{n-1} X_n, n = 1,2,...\}$ forms a renewal process. It can be shown that if $\{X_n, n = 1,2,...\}$ is a GP and $f(x)$ is the probability density function with mean α and variance σ^2 then $\lambda^{n-1} f(\lambda^{n-1} x)$ will be the probability density function of X_n with mean $E(X_n) = \dfrac{\alpha}{\lambda^{n-1}}$ and $\text{var}(X_n) = \dfrac{\sigma^2}{\lambda^{2(n-1)}}$. Thus, the important parameters of GP are λ, α and σ^2 are to be estimated.

Generalized Exponential Life Distribution

The probability density function (pdf) of a generalized exponential distribution is given by

$$f(x,\alpha,\beta) = \frac{\alpha}{\beta} e^{-\frac{x}{\beta}} \left(1 - e^{-\frac{x}{\beta}}\right)^{\alpha-1}, \alpha, \beta, x > 0 \tag{3.1}$$

where $\alpha,\ \beta > 0$ and are the shape and scale parameter of the life distribution respectively. The above discussed life distribution can be abbreviated as $GE(\alpha,\beta)$ with the shape α and the scale parameter β. If $\alpha = 1$, then it is written as $GE(1,\beta)$ and shows the exponential distribution with scale parameter β. The generalized exponential distribution with two parameters may be used for analyzing lifetime data, particularly, in places of Gamma and Weibull distributions with two parameters. Its shape parameter depicts the behavior of failure rate: which may be increasing or decreasing depending on the values of shape parameters. The distribution function of life distribution of the items is given as follows

$$F(x,\alpha,\beta) = \left(1 - e^{-\frac{x}{\beta}}\right)^{\alpha}, \alpha, \beta, x > 0 \tag{3.2}$$

The survival function of the items takes the following form

$$S(x,\alpha,\beta) = 1 - \left(1 - e^{-\frac{x}{\beta}}\right)^{\alpha} \tag{3.3}$$

The hazard rate function is given as follows

$$h(x,\alpha,\beta) = \frac{\frac{\alpha}{\beta} e^{-\frac{x}{\beta}} \left(1 - e^{-\frac{x}{\beta}}\right)^{\alpha-1}}{1 - \left(1 - e^{-\frac{x}{\beta}}\right)^{\alpha}} \tag{3.4}$$

The shape of the failure rate function depends only on the values of parameter α . For $\alpha > 1$, the GE distribution has a log concave density and for $\alpha \le 1$ it has log-convex. Therefore, when $\alpha > 1$ and β is constant, then it has an increasing failure rate and when $\alpha < 1$ with constant β , then it has decreasing failure rate. The failure rate function of GE distribution and Gamma distribution, with two parameters behaves in same way, while the failure rate function of Weibull distribution behaves in totally different way.

Assumptions and Method of Test

1. The failure time of items follows generalized exponential distribution $GE(\alpha,\beta)$ at all the stresses.
2. Suppose a life test with different S stress level (i.e. increasing stress level) is conducted. Under each stress level, n items from a random sample have been put on test at the same time. Let x_{ki} , $i = 1,2,3...,n$ $k = 1,2,...,s$ be the failure time of i^{th} item under k^{th} stress level. We will remove the failed the items from thetest and it would run till the whole random sample exhausted (complete sample).
3. There is a linear relationship between the log of shape parameter and stress i.e., $\log(\beta_k) = p + qS_k$, where *p* and *q* are unknown, and their values depends on the nature of products and method.
4. The lifetimes of items under each stress level is denoted by the random variables $X_0, X_1, X_2,..., X_S$, where X_0 is the lifetime of the items under normal stress (or design stress) at which the items will run normally and sequence $\{X_k, k = 1,2,...s\}$ forms a GP with parameter $\lambda > 0$.

The assumptions discussed above are very common in ALT literature except the last one, i.e. assumption 4. It is the assumption of geometric process which is simply better among the available usual methods without increasing the level of difficulty in calculations. By assuming the linear relationship between the log of life and stress (i.e. assumption 3), the assumption can be shown by the following theorem.

Theorem 3.1: If the level of stress in an ALT is increasing with a constant difference then under each stress level the lifetimes of items forms a GP, i.e. if $S_{k+1} - S_k$ is constant for $k = 1,2,..., s-1$, then $\{X_k, k = 1,2,..., s\}$ forms a GP.

Proof: We get the following equation from assumption (3)

$$\log\left(\frac{\beta_{k+1}}{\beta_k}\right) = q(S_{k+1} - S_k) = q\Delta S \tag{3.5}$$

From the above equation we can see that the increased stress levels form a sequence with a difference ΔS, this sequence is called arithmetic sequence that is formed with constant difference ΔS.

Here, we can write the equation (3.5) as follows

$$\frac{\beta_{k+1}}{\beta_k} = e^{b\Delta S} = \frac{1}{\lambda}(say) \tag{3.6}$$

It is obvious from equation (3.6) that

$$\beta_k = \frac{1}{\lambda}\beta_{k-1} = \frac{1}{\lambda^2}\beta_{k-2} = ... = \frac{1}{\lambda^k}\beta$$

Therefore, the probability density function of the lifetimes at the kth stress level is given as

$$f_{X_k}(x) = \frac{\alpha}{\beta_k} e^{\frac{-x}{\beta_k}}\left(1 - e^{\frac{-x}{\beta_k}}\right)^{\alpha-1}$$

$$= \frac{\alpha\lambda^k}{\beta} e^{\frac{-\lambda^k x}{\beta}}\left(1 - e^{\frac{-\lambda^k x}{\beta}}\right)^{\alpha-1} \tag{3.7}$$

and the cdf is

$$F_{x_k}(x) = \left(1 - e^{\frac{-\lambda^k x}{\beta}}\right)^{\alpha} \tag{3.8}$$

Eq. (3.7) implies that

$$f_{X_k}(x) = \lambda^k f_{X_0}\left(\lambda^k x\right) \tag{3.9}$$

Now, by the definition of geometric process and the equation (3.6) we can see that if probability density function of the lifetime of the products at usual stress level i.e. X_0 , is $f_{X_0}(x)$, then the probability density function of the lifetimes at the kth stress level i.e. X_k is given by $\lambda^k f_{X_0}\left(\lambda^k x\right),\ k = 1,2,...,s$. Hence, it is obvious that failure times of the products form a geometric process with parameter λ under a sequence of arithmetically increasing stress levels.

Expression (3.7) shows that if the lifetimes of products under a sequence of increasing stress level form a geometric process with ratio λ and if the lifetime of the items at normal stress level follows generalized exponential distribution with characteristic β , then the lifetime distribution of the test items at k^{th} stress level will also be generalized exponential with characteristic life $\dfrac{\beta}{\lambda^k}$.

Estimation Process

Maximum likelihood estimation (MLE) is one of the extensively used methods among all estimation methods. It can be applied to any probability distribution while other methods are somewhat restricted. The use of MLE in ALT is difficult and mathematically very complex and, even most of the times the closed form estimates of parameters do not exist. Therefore, Newton Raphson method is used to estimate the numerical values of them. The likelihood function for constant stress ALT for complete case generalized exponential failure data using GP for s stress levels is given by:

$$L = \prod_{k=1}^{s}\prod_{i=1}^{n}\frac{\alpha\lambda^k}{\beta} e^{-\frac{\lambda^k x_{ki}}{\beta}}\left(1 - e^{-\frac{\lambda^k x_{ki}}{\beta}}\right)^{(\alpha-1)} \tag{3.10}$$

Now take the log of the above function and rewrite as follows;

$$l = \sum\sum\left[\log\alpha + k\log\lambda - \log\beta - \frac{\lambda^k x_{ki}}{\beta} + (\alpha-1)\log(1 - e^{-\frac{\lambda^k x_{ki}}{\beta}})\right] \tag{3.11}$$

The MLEs of the parameters $\alpha,\ \beta$ and λ can be obtained by solving the following normal equations $\dfrac{\partial l}{\partial\alpha} = 0, \dfrac{\partial l}{\partial\beta} = 0$ and $\dfrac{\partial l}{\partial\lambda} = 0$.

$$\frac{\partial l}{\partial \alpha} = \frac{ns}{\alpha} + \sum_{k=1}^{s}\sum_{i=1}^{n}\log(1-z) = 0 \tag{3.12}$$

$$\frac{\partial l}{\partial \beta} = \frac{-ns}{\beta} + \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{\lambda^k x_{ki}}{\beta^2} - \frac{\lambda^k x_{ki}}{\beta^2} \times (\alpha-1)\frac{Z}{(1-Z)} = 0\right] = 0 \tag{3.13}$$

$$\frac{\partial l}{\partial \lambda} = \frac{kns}{\lambda} - \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{k\lambda^{(k-1)}x_{ki}}{\beta} - k\lambda^{k-1}x_{ki}\,\frac{(\alpha-1)}{\beta}\frac{Z}{1-Z}\right] = 0 \tag{3.14}$$

where $Z = e^{-\frac{\lambda^k x_{ki}}{\phi}}$

We use above equations namely (3.12), (3.13) and (3.14) to find the estimate of α, β and λ.

Fisher Information Matrix and Asymptotic Confidence Interval

The asymptotic Fisher Information matrix is given by:

$$\begin{bmatrix} -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \beta} & -\frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 l}{\partial \beta \partial \alpha} & -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 l}{\partial \lambda \partial \alpha} & -\frac{\partial^2 l}{\partial \beta \partial \lambda} & -\frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}$$

The elements of Fisher Information matrix can be obtained by putting a negative sign before double and partial derivatives of the parameters, which are given as follows;

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{ns}{\alpha^2}$$

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{ns}{\beta^2} - \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{2\lambda^2 x_{ki}}{\beta^3} + (\alpha-1)\lambda^2 x_{ki}\left\{\frac{Z\lambda^k x_{ki} - 2Z\beta(1-Z)}{\beta^4(1-Z)}\right\}\right]$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{kns}{\lambda^2} - \sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{k(k-1)\lambda^{k-2}x_{ki}}{\beta} - \frac{kZ(\alpha-1)x_{ki}}{\beta}\left\{\frac{(k-1)\lambda^{k-2}}{(1-Z)} - \frac{k\lambda^{2(k-1)}x_{ki}}{\beta(1-Z^2)}\right\}\right]$$

$$\frac{\partial^2 l}{\partial\alpha\,\partial\beta}=\frac{\partial^2 l}{\partial\beta\,\partial\alpha}=-\sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{Z\lambda^k x_{ki}}{\beta^2(1-Z)}\right]$$

$$\frac{\partial^2 l}{\partial\alpha\,\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\,\partial\alpha}=\sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{kZ\lambda^{k-1} x_{ki}}{\beta(1-Z)}\right]$$

$$\frac{\partial^2 l}{\partial\beta\,\partial\lambda}=\frac{\partial^2 l}{\partial\lambda\,\partial\beta}=\sum_{k=1}^{s}\sum_{i=1}^{n}\left[\frac{k\lambda^{k-1}x_{ki}}{\beta^2}-\frac{(\alpha-1)Zkx_{ki}}{\beta^2}\left\{\frac{\lambda^{k-1}}{(1-Z)}-\frac{\lambda^{2k-1}x_{ki}}{\beta(1-Z)^2}\right\}\right]$$

The confidence interval for parameters α, β and λ are given as follows:

$\hat{\alpha}+Z_{1-\frac{\phi}{2}}\left(SE(\hat{\alpha})\right), \hat{\beta}+Z_{1-\frac{\phi}{2}}\left(SE(\hat{\beta})\right)$, and $\hat{\lambda}+Z_{1-\frac{\phi}{2}}\left(SE(\hat{\lambda})\right)$ respectively.

Simulation Study

Simulation is used for observing the statistical properties of parameters. It is an attempt to model an assumed condition to study the behaviour of function.

To perform the study, we first generate a random sample from Uniform distribution by using optim() function in R software.

Now we use inverse cdf method to transform equation (3.8) in terms of *u* and get the expression of x_{ki} , k=1,2,3,...,s and i=1,2,3,...,n.

$$x_{ki}=-\beta\frac{(1-u^{1/\alpha})}{\lambda^k},\quad k=1,2,\cdots,s\;\; i=1,2,\cdots,n$$

- Now take the random samples of size 20,40,60,80 and 100 from the generalized exponential distribution and replicate them 1000 times.
- The values of parameters and numbers of the stress levels are chosen to be $\alpha=1.2,\beta=2.8,\lambda=1.1,s=4 or 6$.
- optim() function is used to obtain the ML estimates, relative absolute bias(RAB), the mean squared error(MSE), relative error(RE) and lower and upper bound of 95% and 99% confidence intervals for different sample sizes.

The outcome obtained in the above study are summarized in Table3.1 and Table 3.2.

Table 3.1: *Simulation results of Generalised exponential using GP at* $\alpha = 1.2, \beta = 2.8,$ $\lambda = 1.1, s = 4$

Sample	Estimates	Mean	SE	√MSE	RABias	RE	Lower BOund	Upper Bound
20	α	1.0772	0.1990	0.1923	0.1022	0.1602	0.6870 0.5636	1.4674 1.5908
	β	3.0781	0.3191	0.0950	0.0993	0.0339	2.4526 2.2547	3.7036 3.9015
	λ	1.0500	0.1034	0.0999	0.0908	0.0864	0.7971 0.7329	1.2028 1.2670
40	α	1.1675	0.1200	0.1159	0.02707	0.0966	0.9322 0.8578	1.4027 1.4771
	β	3.0397	0.2565	0.0614	0.0856	0.0219	2.5369 2.3779	3.5424 3.7015
	λ	0.9952	0.1035	0.1000	0.0909	0.0558	0.7970 0.7329	1.2028 1.2670
60	α	1.2113	0.1201	0.1160	0.0094	0.0967	0.9758 0.9014	1.4467 1.5212
	β	3.0038	0.2417	0.0545	0.0728	0.0194	2.5299 2.3800	3.4777 3.6276
	λ	1.0000	0.1034	0.0999	0.0908	0.0495	0.7971 0.7330	1.2028 1.2670
80	α	1.1826	0.0805	0.0777	0.0144	0.0648	1.0248 0.9749	1.3404 1.3904
	β	3.0491	0.2645	0.0653	0.0889	0.0233	2.5306 2.3666	3.5676 3.7316
	λ	0.9924	0.1035	0.1000	0.0909	0.0593	0.7970 0.7328	1.2029 1.2671
100	α	1.1579	0.0867	0.0838	0.0350	0.0698	0.9878 0.9340	1.3280 1.3818
	β	3.0515	0.2661	0.0661	0.0898	0.0236	2.5299 2.3649	3.5731 3.7381
	λ	1.0041	0.1035	0.0999	0.0909	0.0600	0.7971 0.7329	1.2028 1.2670

Table 3.2: *Simulation results of Generalised exponential using GP at* $\alpha = 1.2, \beta = 2.8, \lambda = 1.1, s = 6$.

Sample	Estimates	Mean	SE	√MSE	RABias	RE	Lower BOund	Upper Bound
20	α	1.2838	0.1409	0.1361	0.0698	0.1134	1.0076 0.9202	1.5601 1.6474
	β	2.9798	0.2183	0.0444	0.0642	0.0158	2.5518 2.4165	3.4077 3.5431
	λ	0.9950	0.1035	0.1000	0.0909	0.0404	0.7970 0.7328	1.2029 1.2671
40	α	1.1598	0.1451	0.1402	0.0334	0.1168	0.8753 0.7853	1.4442 1.5342
	β	3.0598	0.2899	0.0784	0.0928	0.0280	2.4916 2.3118	3.6281 3.8079
	λ	1.0489	0.1035	0.1000	0.0909	0.0713	0.7971 0.7329	1.2028 1.2670
60	α	1.2019	0.0968	0.0935	0.0015	0.0779	1.0120 0.9519	1.3917 1.4517
	β	3.0650	0.2777	0.0719	0.0946	0.0257	2.5207 2.3485	3.6092 3.7814
	λ	0.9923	0.1035	0.1000	0.0909	0.0654	0.7970 0.7329	1.2028 1.2670
80	α	1.1868	0.0892	0.0861	0.0110	0.0718	1.0119 0.9566	1.3616 1.4169
	β	3.0600	0.2708	0.0684	0.0928	0.0244	2.5291 2.3612	3.5909 3.7588
	λ	0.9973	0.1035	0.1000	0.0909	0.0622	0.7970 0.7328	1.2029 1.2671
100	α	1.1977	0.0812	0.0785	0.0019	0.0654	1.0384 0.9880	1.3570 1.4073
	β	3.0587	0.2718	0.0689	0.0923	0.0246	2.5260 2.3574	3.5914 3.7600
	λ	1.0500	0.1035	0.1	0.0909	0.0626	0.7971 0.7329	1.2028 1.2670

Conclusion

In this study, geometric process is introduced for the study of accelerated life testing plan under constant stress when the failure time data are from a generalized exponential model. It is better choice for life testing because of its simplicity in nature. The mean, SE, MSE, RAB and RE of the parameters are obtained and based on the asymptotic normality, the 95% and 99% confidence intervals of the parameters are also obtained. The outcome in Table 3.1 and Table 3.2 show that the estimated values of α , β and λ are very close to true (or initial) values with very small SE and MSE. As sample size increases, the value of SE and MSE decreases and the confidence interval become narrower. For the Table 3.2, the maximum likelihood estimators have good statistical properties than the Table 3.1 for all sample size. The future research should extend the GP model into ALTg for different life distribution. Introducing the GP model into ALT with other test plans or censoring techniques is another object of the future research.

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How to Cite this Chapter:

Rahman, Ahmadur, and Showkat Ahmad Lone. "Mathematical Model of Accelerated Life Testing Plan Using Geometric Process." *Bayesian Analysis and Reliability Estimation of Generalized Probability Distributions*, edited by Afaq Ahmad, AIJR Publisher, 2019, pp. 30–40, ISBN: 978-81-936820-7-4, DOI: 10.21467/books.44.3.

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