On (i, j) β^*-Closed Sets in Bitopological Spaces

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ABSTRACT

In this paper, we introduce and study the concept of a new class of closed sets called $(i,j)\ \beta^*$ $(i,j)\ \beta^*$-closed sets (briefly β^*-closed set) in Bitopological spaces. Also, we investigate some of their properties.

Keywords: generalized closed sets, bitopology applications.

1 Introduction

A bitopological space (X, $|_1$, $|_2$) defined to be a set X equipped with two topologies $|_1$, $|_2$ on X introduced by JC. Kelly in 1963 and he initiated a systematic study of bitopological space. The study of generalized closed sets in a bitopological space was initiated by Levine in [8] and the concept of $T_{1/2}$ spaces was introduced. Various authors have turned attention to the various concepts of topology by considering bitopological spaces instead of topological spaces. In 2008, S Jafari, T. Noiri, N. Rajesh and M.L. Thivagar [6] introduced the concept of $\tilde{g}$-closed sets and discussed some of their properties. The notion of β-open sets and β-continuity introduced by Abd El-Monsef et al. [1] in topological spaces. Further continuous β*-open sets in Topological spaces introduced by Mubaraki [9].

The purpose of this paper Introduce and study the notions of (i, j)-β*- closed sets and (i, j)-β*-open sets in bitopological space and (β*, $|_1$, $|_2$)-graph by utilizing the notion of (i, j)-β*-closed set. And also, some of characterizations and properties of these notions are investigated.

2 Preliminaries

Definition 2.1. Let *X be a set. B* is a subset of a space X. then

(1) α-open [8] if $B \subseteq\subseteq int(cl(int(B)))$,

(2) preopen [5] if $B \subseteq\subseteq int(cl(B))$,

(3) δ-preopen [9] if $B \subseteq\subseteq int(cl_\delta(B))$,

(4) β- open set [1]t if $B \subseteq\subseteq cl(int(cl\ B))$

(5) b- open set [10] if $B \subseteq\subseteq cl(int(B)) \cup\cup int(cl\ B))$

(6) regular open [5] if B = int(cl(B))

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The complement of a α -open (resp. preopen, δ-preopen, β-open, *b*-open, regular open) sets is called a α -closed [4] (resp. preclosed, δ-pre-closed, β-closed, *b*-closed, regular closed).

Definition 2.2. Let *X be a set. B* is a subset of a space (X, $|_i$, $|_j$). then B is said to be

(a) (i,j)-pre-open (briefly, ij-p-open)[5] B ⊆⊆ $|_i$-int ($|_j$ –cl(B)),

(b) (i,j)-regular-open (briefly, ij-r-open) [5] B = $|_i$ -int ($|_i$ –cl(B)),

(c) (i,j)-regular-closed (briefly, ij-r-closed)[3] B = $|_j$-cl ($|_i$ –int(B)),

(d) (i,j)-semi-open (briefly, ij-s-open)[2] B ⊆⊆ $|_j$ -cl ($|_i$ –int(B)),

(e) (i,j)-generalized closed (briefly, ij-g-closed) B ⊆⊆U & U ∈ $|_i$ ⇒ $|_j$ –cl(B) ⊆ ⊆ U,

(f) (i,j)−regular generalized closed (briefly, ij−rg−closed) B≤U & U ∈ ij-RO(X)

⇒ $|_j$ −cl(B)≤ U,

(g) (i,j)-〈 open set if B ≤ $|_i$ –int ($|_j$ -cl ($|_i$ –int(B)))

(h) (i,j)−regular weakly closed(briefly, ij−rw−closed) set if B≤U & U ∈ ij-RSO(X)

⇒ $|_j$ −cl(B)≤ U,

Naturally, the complement of respective open/closed set is respective closed/open set.

The class of ij-k-open (resp. ij-k -closed) sets is denoted by the symbol ij-kO(X)(resp. ij-kC(X)) where k = P,R,S,G,RG, 〈, RW accordingly.

Lemma 2.3. [1, 9,10]. Let B be a subset of a space (X, τ_1, τ_2). Then:

(1) ij-δ-pint(B) = B ∩ τ_i-int (τ_j -cl$_\delta$(B)) and ij-δ-pcl(B)= B ∪ τ_j-cl(τ_i -intδ(B)),

(2) ij-β-int(B) = B ∩ τ_j -cl (τ_i -int (τ_j -cl(B))) and ij-β-cl(B)= B ∪ τ_i -int(τ_j -cl(τ_i -int(B))).

Definition 2.4. [10] Let B be a subset of a space (X, τ_1, τ_2). Then:

(a) ij−b−open set if B ≤ τ_j -pcl (τ_i –pint(B)),

(b) ij-b-closed set if τ_i -pint (τ_j −pcl(B)) ≤ B or B^c ij-b- open.

Symbols ij-BO(X) & ij-BC(X) stand for the class of all ij-b-open & ij-b-closed sets respectively.

Definition 2.5. [10] For any bitopological space (X, τ_1, τ_2) and B ≤ X, then ij−b−interior and ij−b−closure of B are denoted by ij−bint(B) & ij−bcl(B) respectively and defined by:

(a) ij-bint(B) = ∪ { F≤ : F∈ ij−BO(X), F≤ B}

(b) ij−bcl(B) = ∩ { F≤ X : F∈ ij−BC(X), B≤ F}

3 On (i, j)- $\beta^*\beta^*$ Closed sets and (i, j)- $\beta^*\beta^*$ Open sets in Bitopological Spaces

Definition 3.1. For any subset *B* of a topological space (X, τ_1, τ_2) is said to be:

(1) a (i, j) β*-closed set τ_i-*int* (τ_j-*cl* (τ_i-*int* (*B*))) ⊆ U whenever B⊆ U, where U τ_i-$\beta\beta$ open set

(2) a (i, j) β*-open set if the complement of (i, j) β*-closed set is open.

The family of all (i, j)-β*-closed (resp. (i, j) β*-open) subsets of a space (X, τ_1, τ_2) will be as always denoted by β**C*(*X*)(resp. β**O*(*X*)).

Example 3.2. Let X = {p, q, r, s}, τ_1 = {X, ϕ, {p},{r}, {p, q}, {p, r}, {p, q, r}, {p, r, s}} and

τ_2 ={X, ϕ, {p}, {p, q}} .

Then the (i, j) - β*-closed sets are {X, ϕ, {q}, {r}, {s}, {q, r}, {q, s}, {r, s}, {q, r, s}}.

Remark 3.3. If τ_1 = τ_2 = τ in the Definition 3.1., then (i, j) - β*-closed set is a β*-closed in a topological space.

Theorem 3.4. If B is a τ_j -closed subset of (X, τ_1, τ_2) then B is (i, j) - β*-closed set.

Proof: Let B be a τ_j -closed set in (X, τ_1, τ_2). Let G be a τ_i - β -open

set in (X, τ_1, τ_2). Such that B ⊆ G. Then τ_i - int(τ_j - cl(τ_i - int(A))) ⊆ G as B is τ_j -closed

set. This implies τ_j - cl(B) = B ⊆ G. This implies τ_j - cl(B) ⊆ G.

Therefore, B is (i, j) - β*-closed set in (X, τ_1, τ_2) .

The converse of the above theorem need not be true as seen from the following example.

Example 3.5. Let X = {p, q, r}, τ_1 = {X, ϕ, {p}, {q}, {p, q}} and

τ_2= {X, ϕ, {r}, {q, r}, {r, s}, {q, r, s}}.

(i, j) - β*-closed set : {X, ϕ, {p}, {r}, {s}, {p, q}, {p, r}, {p, s}, {q, r}, {r, s}, {p, q, r}, {p, q, s}, {p, r, s}, {q, r, s}} .

τ_2- closed sets: {X, ϕ, {p}, {p, q}, {p, s}, {p, q, s}}

The subset {p, r} is (i, j) - β*-closed set but not τ_2- closed set in the bitopological space (X, τ_1, τ_2).

Theorem 3.6. Every (i, j) - β*-closed set is ($\tau\tau_i$, $\tau\tau_j$)-g-closed.

Proof. Let B be any (i, j) - β*-closed set in X. Let B ⊆ U and U be β -open in X. Every open set is g-open and thus B is (i, j) - β*-closed set. Therefore $\tau\tau_2$-cl(B) ⊆ U. Hence B is ($\tau\tau_i$, $\tau\tau_j$)-g-closed.

The converse of the above theorem need not be true as it is seen from the following example.

Example 3.7. Let X = {p, q, r, s}, τ_1 = {X, ϕ, {p}, {q}, {p, q}} and

τ_2= {X, ϕ, {r}, {q, r}, {r, s}, {q, r, s}}.

(i, j) - β*-closed set : {X, ϕ, {p}, {r}, {s}, {p, q}, {p, r}, {p, s}, {q, r}, {r, s}, {p, q, r}, {p, q, s}, {p, r, s}, {q, r, s}} .

Then the set B = {*q*} *is* (τ_i, τ_j)-*g-closed but not* (i, j) - β*-closed set *in* (*X*,τ_1, τ_2).

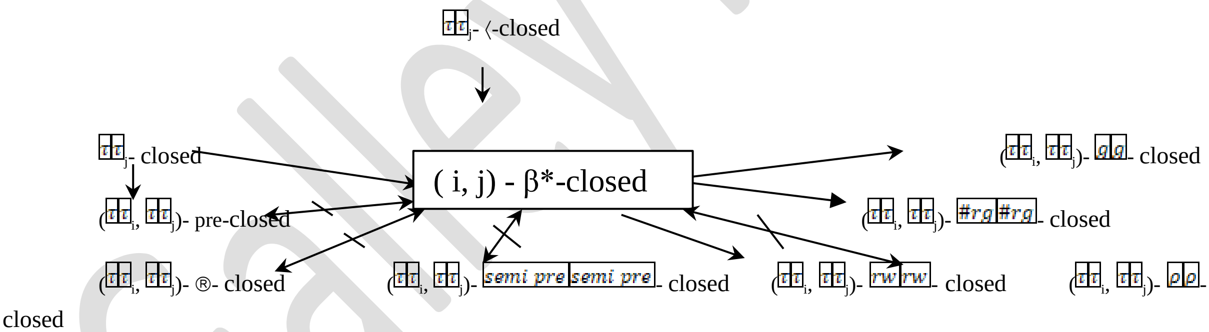
Theorem 3.8. *Every* (i, j) - β*-closed *set is* (τ_i, τ_j)-rw-*closed.*

Proof. Let *B* be any (i, j) - β*-closed set. Let *B* ⊆ *U* and *U* be j - β-open.

Observe that every j - β-open set is open and every open set is τ_1- regular semi open and therefore B is (τ_i, τ_j)-rw-*closed.* It follows that τ_2- *cl*(*B*) ⊆ *U.*

Hence *B* is (τ_i, τ_j)-rw-closed.

Remark 2.1. The following diagram holds for each a subset *B* of *X.*



None of these Implication is reversible.

4 Some Properties of (i, j)- β^* Closed sets and (i, j)- β^* open sets in Bitopological Spaces

Theorem 4.1. Let (*X*, τ_1, τ_2) be a Bitopological spaces. Then the following are hold.

(1) The arbitrary union of (i, j)- β*-open sets is (i, j)- β*-open,

(2) The arbitrary intersection of (i, j)-β*-closed sets is (i, j)- β*-closed.

Proof.

(1) Let {Bi, i ∈ I} be a family of (i, j)-β*-opensets. Then B_i⊆ τ_j -cl(τ_i -int(τ_j -cl(Bi))) ∪ τ_j int(τ_i -cl$_\delta$

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$(B_i))$ and hence $\cup_i B_i \subseteq \cup_i (\tau\tau_j\text{- cl}(\tau\tau_i\text{ -int}(\tau\tau_j\text{ -cl}(B_i))) \cup \tau\,\tau_i\text{ -int}(\tau\tau_j\text{ -cl}_\delta(B_i))) \subseteq \tau\tau_{j}\text{- cl}(\tau\tau_{i}\text{- int}(\tau\tau_j\text{ –cl}(\cup_i B_i))) \cup \tau\,\tau_i\text{ –int }(\tau\tau_{j}\text{- cl}_\delta(\cup_i B_i))$, for all i ∈ I. Thus $\cup_i B_i$ is (i, j)- β*-open,

(2) It follows from (1).

Remark 4.2. By the following example we show that the intersection of any two β*-open sets is not β*-open.

Example 4.3. Let X = {p, q, r, s}, $|_1$ = {X, ϕ, {p}, {q}, {p, q}} and

$|_2$= {X, ϕ, {r}, {q, r}, {r, s}, {q, r, s}}.

(i, j) - β*-closed set: {X, ϕ, {p}, {r}, {s}, {p, q}, {p, r}, {p, s}, {q, r}, {r, s}, {p, q, r}, {p, q, s}, {p, r, s}, {q, r, s}}.

(i, j) - β*-open set: {X, ϕ, {p}, {q}, {r}, {s}, {p, q}, {p, s}, {q, r}, {q, s}, {r, s}, {p, q, r}, {p, r, s}, {q, r, s}.

Then *A* = {*p, q, r*} and *B* = {p, *r, s*} are (i, j)- β*-open sets. But, *A* ∩ *B* = {p, *r*} is not (i, j)-β*-open.

Definition 4.4. Let $(X, \tau\,\tau_1, \tau\tau_2)$ be a bitopological space. B is subset of X, then:

(a) (i, j)- β*-*int*(*B*) = ∪ {F ≤ X: F∈ ij−BO(X), F≤ B}

(b) (i, j)- β*-cl(B) = ∩ {F≤ X: F∈ ij−BC(X), B≤ F}

Theorem 4.5. Let A, B be two subsets of a bitopological space $(X, \tau\,\tau_1, \tau\tau_2)$. Then the following are hold:

(1) $|_2$-β*-cl(X) = X and $|_2$- β*-cl(φ) = φ,

(2) A ⊆ $|_2$-β*-cl(A),

(3) If A ⊆ B, then $|_2$-β*-cl(A) ⊆ $|_2$-β*-cl(B),

(4) x ∈ $|_2$-β*-cl(A) if and only if for each a $|_1$-β*-open set U containing x, U ∩ A $\bar{\ }$ φ,

(5) A is (i, j)-β*-closed set if and only if A = $|_2$-β*-cl(A),

(6) $|_2$-β*-cl ($|_2$ -β*-cl(A)) = $|_2$ - β*-cl(A),

(7) $|_2$ -β*-cl(A) ∪ $|_2$ - β*-cl(B) ⊆ $|_2$ -β*-cl (A ∪ B),

(8) $|_2$-β*-cl (A ∩ B) ⊆ $|_2$ -β*-cl(A) ∩ $|_2$ -β*-cl(B).

Proof. (1) Trivial case. That is cl(X)= X and Cl(φ)= φ => $|_2$-β*-cl(X) = X and $|_2$- β*-cl(φ) = φ

(2) By known result int A □ A □ cl(A) => A ⊆ $|_2$-β*-cl(A)

(3) By (2) A ⊆ $|_2$-β*-cl(A) and B ⊆ $|_2$-β*-cl(B). Given A ⊆ B => $|_2$-β*-cl(A) ⊆ $|_2$-β*-cl(B)

(4) x ∈ $|_2$-β*-cl(A) if and only if for each a $|_1$-β*-open set U containing x, U ∩ A $\bar{\ }$ φ,

To prove that contra positive. If x ∉∉ $|_2$-β*-cl(A) <=> there exist an open set containing x does not intersect A.

=> If x $\notin\notin$ $|_2$-β*-cl(A) then the set U= X - $|_2$-β*-cl(A) is an open set containing x does not intersect A.

<= If there exist an open set containing x does not intersect A. Then X- U is a closed set containing A.

By the definition of β*-closure, the set X-U must contain A. Therefore x $\notin\notin$ $|_2$-β*-cl(A)

(5) A is (i, j)-β*-closed set if and only if A = $|_2$-β*-cl(A).

We know that, A is open => A= int (A) and A is closed then A = Cl(A) in topological spaces.

⇨ A is (i, j)-β*-closed set A = $|_2$-β*-cl(A)

(6) By using (2) and A ⊆ $|_2$-β*-cl(A), we have $|_2$- β*-cl(A) ⊆ $|_2$- β*-cl($|_2$-β*-cl(A)). Let x ∈ $|_2$- β*-cl($|_2$-β*-cl(A)). Then, for every $|_1$-β*-open set V containing x, V ∩ $|_2$-β*-cl(A) $\bar{}$ φ.

Let y ∈ V ∩ $|_2$-β*-cl(A). Then, for every $|_1$-β*-open set G containing y, A ∩ G $\bar{}$ φ. Since V is a $|_1$-β*-open set, y ∈ V and A ∩ V $\bar{}$ φ, then x ∈ $|_2$-β*-cl(A).

Therefore, $|_2$-β*-cl($|_2$-β*-cl(A)) ⊆ $|_2$-β*-cl(A).

Theorem 4.6. Let A, B be two subsets of a bitopological space (X, $\tau\tau_1$, $\tau\tau_2$). Then the following are hold:

(1) $\tau\tau_1$-β*-int(X) = X and $\tau\tau_1$-β*-int(φ) = φ,

(2) $\tau\tau_1$-β*-int(A) ⊆ A,

(3) If A ⊆ B, then $\tau\tau_1$-β*-int(A) ⊆ $\tau\tau_1$-β*-int(B),

(4) x ∈ $\tau\tau_1$-β*-int(A) if and only if there exist $\tau\tau_1$-β*-open W such that x ∈ W ⊆ A,

(5) A is $(i,j)(i,j)$-β*-open set if and only if A = $\tau\tau_1$-β*-int(A),

(6) $\tau\tau_1$-β*- $\tau\tau_1$-int ($\tau\tau_1$-β*-int(A)) = $\tau\tau_1$-β*-int(A),

(7) $\tau\tau_1$-β*-int (A ∩ B) ⊆ $\tau\tau_1$-β*-int(A) ∩ $\tau\tau_1$- β*-int(B),

(8) $\tau\tau_1$-β*-int(A) ∪ $\tau\tau_1$-β*-int(B) ⊆ $\tau\tau_1$-β*-int (A ∪ B).

Proof: By using above theorem, it is obvious.

Theorem 4.7. for an $(i,j)(i,j)$-β*-closed and $\tau\tau_i$-open set A in a bitopological space (X, $\tau\tau_1$, $\tau\tau_2$), the set B ≤ A is $(i,j)(i,j)$-β*-closed relative to A If B is $(i,j)(i,j)$-β*-closed in X.

Proof: Since, A is both $(i,j)(i,j)$-β*-closed and $\tau\tau_i$-open set in a bitopological space, hence, (i, j)-β*cl(A) ≤ A. Also, B ≤ A provides that (i, j)- β*cl(B) ≤ (i, j)- β*cl(A). Combining these facts, we have (i, j)-β*cl(B) ≤ (i, j)- β*cl(A) ≤ A.

Now, A ∩ (i, j)− β*cl(B) = (i, j)− β*cl_A(B). Using it, we get (i, j)- β*cl_A(B) = (i, j)- β*cl(B) ≤ A.

If B is (i, j)- β-closed relative to A and U is $\tau\tau_i$ −open set in X such that B ≤ U, then B = B∩A ≤ U ∩A where U∩A is $\tau\tau_{i\,A}$-open (or i-open in A).

Hence as B is ij-gb-closed relative to A, (i, j)- β*-cl(B) = (i, j)- β*cl_A(B) ≤

U ∩ A ≤ U. Consequently, B is (i, j)− β*closed in X.

Conversely, if B is (i, j)- β*-closed in X and U is an $\tau\tau_i$ −open subset of A such that B≤ U, then U = V∩A for some $\tau\tau_i$ −open subset V of x. As B≤ V and B is (i, j)− β*−closed set in X, (i, j)− β*cl(B) ≤ V. Thus, (i, j)− β*cl_A(B) = (i, j)- β*cl(B) ∩ A ≤ V∩ A = U. Consequently, B is (i, j)- β*-closed relative to A.

Corollary 4.8. If A is an (i, j)- β*-closed & $\tau\tau_i$ -open set in a bitopological space (X, $\tau\tau_1$, $\tau\tau_2$) then A ∩ F is also (i, j)− β*−closed whenever F ∈ (i, j)- β*C(X).

Proof: Let A be (i, j)- β*-closed & $\tau\tau_i$ -open set in a bitopological space (X, $\tau\tau_1$, $\tau\tau_2$). For A to be (i, j)- β*-closed as well as $\tau\tau_i$ –open, it is natural that (i, j)- β*cl(A) ≤ A. So, A is (i, j)- β*closed.

Again, as F ∈ (i, j)- β*C(X) & A ∈ (i, j)- β*C(X) so A∩ F ∈ (i, j)- β*C(X). Now, A∩ F ≤ A $|_j$- β*cl (A∩ F) ≤ A which means that A ∩ F is (i, j)- β*-closed.

Theorem 4.9. If A is an (i, j)- β*-closed set and B is any set such that A ≤ B≤ (i, j)- βcl(A), then B is also an (i, j)- β*-closed set.

Proof: Let B ≤ U where U is $\tau\tau_i$ -open in (X, $\tau\tau_1$, $\tau\tau_2$). Since, A is (i, j)- β*-closed and A ≤ U, then (i, j)- βcl(A) ≤U.

Again, A ≤B≤ (i, j)− βcl(A) □ (i, j)- βcl(A) = (i, j)- βcl(B). Therefore, combining these facts, we conclude that (i, j)- βcl(B) ≤U whenever B ≤ U & U is $\tau\tau_i$ -open. So, B is also an (i, j)- β*-closed set.

5 Declarations

5.1 Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Abd El-Monsef, M.E., El-Deeb, S.N. and Mahmoud, R.A, "β-open sets and β-continuous mappings", Bull. Fac. Sci. Assiut Univ. vol. 12, pp. 77–90, 1983.

[2] D. Andrijevic. "Semi- preopen sets", Mat. Vesnik, vol. 38, no. 1, pp. 24-32, 1986.

[3] S.S. Benchalli and R. S. Wali, "On rw –closed sets in topological spaces" Bull. Malays. Math., Sci. Soc. vol. 30 (22), pp. 99- 110, Jan. 2007.

[4] T. Fukutake,P. Sundaram and M. Shaik John, "w- closed sets, w- open sets and w- continuity in Bitopological spaces", Bull. Fukuoka Univ. Ed. Part III, vol. 51, pp. 1- 9, 2002.

[5] Y. Gnanambal. "On Generalized Pre regular closed sets in topological spaces", Indian J. Pure Appl. Maths., vol. 28 (3), pp. 351-360, 1997.

[6] S. Jafari, T. Noiri, N. Rajesh and M.L. Thivagar. "Another generalization of closed sets", Kochi J.Math, vol. 3, pp. 25-38, 2008.

[7] J.C.Kelly, "Bitopological Spaces", Proc. Londan Math. Soc., vol. 13, pp. 71 -81, 1963, https://doi.org/10.1112/plms/s3-13.1.71.

[8] N. Levine. “Semi-open sets, semi-continuity in topological spaces”, Amer Math, Monthly, vol. 70, pp. 36-41, Mar. 1963, https://doi.org/10.1080/00029890.1963.11990039.

[9] Ali M. Mubarki, Massed M. Al-Rshudi & Mohammad A. Al-Juhani, “β*-Open sets and β*-continuity in topological spaces”, Journal of Taibah University for Science, vol. 8, pp. 142–148, Apr. 2014, https://doi.org/10.1016/j.jtusci.2013.09.006.

[10] Dr. Thakur C. K. Raman, “On Generalized b-Closed Sets in Bitopological Spaces”, International j. of Scientific Engineering and Technology, vol. 5(2), pp. 124-127, 2016.