Upper Square Free Detour Number of Graphs

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ABSTRACT

In this article we introduce the minimal square free detour sets and investigate the upper square free detour number of a graph. A square free detour set of vertices in a graph is called a minimal square free detour set if no proper subset is a square free detour set of the graph. The upper square free detour number is the maximum order of its minimal square free detour set of the graph. We also determine the upper square free detour number of standard graphs that supports robotics path planning and biological network analysis for efficient navigation.

Keywords: Square free detour distance, Square free detour number, Upper square free detour number.

1 Introduction

In this article, a graph $G = (V, E)$ is considered to be a finite, connected and undirected graph of order $n(n \geq 2)$ with neither loops nor multiple edges. Let $D(x, y)$ be the longest path in G and an $x - y$ path of $D(x, y)$ is called $x - y$ detour. This detour concept was studied by Chartrand [1,2]. The detour concept was generalised to connected concepts and triangle free detour concept by S. Athisayanathan et al. [4,6]. A triangle free detour number of G denoted by $dn_{\Delta f}(G)$ is defined as the minimum order of triangle free detour set S consisting of every pair of vertices of all the triangle free detours in which every vertex of G lies on. A triangle free detour set S is called minimal triangle free detour set if no proper subset of S is a triangle free detour set of G. The upper triangle free detour number of G is the maximum order of its minimum triangle free detour sets and denoted by $dn^{+}_{\Delta f}(G)$. This concept was studied by S. Athisayanathan and S. Sethu Ramalingam in [7,8].

In this article, we extend the triangle free detour distance to square free detour distance denoted by $D_{\square f}(x, y)$ where $x, y \in G$. We define a square free detour set to be a set S in G such that every vertex of G lies on a square free detour joining a pair of vertices of S. Minimum order of square free detour sets of G is called the square free detour number and denoted by $dn_{\square f}(G)$ [5]. Also, we introduce the minimal square free detour set which leads to the upper square free detour number of a graph G. Upper square free detour numbers have significant applications in various fields. In graph theory, detour numbers facilitate the analysis of graph structures, particularly in network optimization problems. This concept extends to network optimization, where upper square-free detour numbers enable the identification of optimal paths and routes in communication networks, transportation systems, and logistics. Furthermore, these numbers play a crucial role in coding theory, contributing to the development of error-correcting codes that ensure reliable data transmission.

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The following theorems are used in the results of this article. For the basic terminologies we refer to Chartrand [2].

Theorem 1.1 [3] Every end-vertex of a non-trivial connected graph G belongs to every square free detour set of G. Also, if the set S of all end-vertices of G is a square free detour set, then S is the unique square free detour basis for G.

Theorem 1.2 [3] If T is a tree with k end-vertices, then $dn(T) = k$.

Theorem 1.3 Let $G = (V, E)$ be a complete bipartite graph $K_{m,n} (2 \leq m \leq n)$ with partitions X and Y where $|X| = m, |Y| = n$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if $S = X$.

Theorem 1.4 Let $G = (V, E)$ be an odd cycle C_n of order $n \geq 3$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of any two adjacent vertices of G.

Theorem 1.5 Let $G = (V, E)$ be an even cycle C_n of order $n \geq 6$. Then a set $S \subseteq V$ is a square free detour basis of G if and only if S consists of any two adjacent vertices or two antipodal vertices of G.

Definition 1.6 Antipodal vertices are pairs of vertices in a graph that are opposite each other in some sense. Independent non-antipodal vertices are vertices in a graph that are not adjacent to each other, not antipodal, and do not have any common neighbors.

2 Upper Square free detour number

Definition 2.1 Let $G = (V, E)$ be a connected graph. A square free detour set S of G is called a *minimal square free detour set* if no proper subset of S is a square detour set of G. The *upper square free detour number* $dn^{+}_{\square f}(G)$ is the maximum order of its minimal square free detour set of G.

Example 2.2 For the graph G given in Figure 2.1, the sets $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_3, v_4, v_5, v_6\}$ are the square free detour set and minimal square free detour set of G respectively. Hence the upper square free detour number of G is 4.

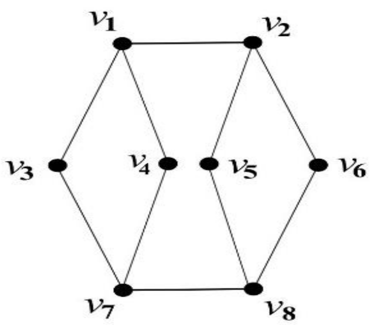

Figure 2.1: G

Remark 2.3 Every minimum square free detour set is a minimal square free detour set, but the converse need not to be true. For the graph G given in Figure 2.1, S_2 is the minimal square free detour set but not a minimum square free detour set of G.

Since every minimal square free detour set of G is a square free detour set of G, we have the following theorem.

Theorem 2.4 For any connected graph G, $dn_{\square f}(G) \leq dn^{+}_{\square f}(G)$.

Proof. Let S be any square free detour basis of G. Then S is also a minimal square free detour set of G and hence the result follows.

Remark 2.5 The bound in Theorem 2.4 is sharp. For any path P_n and complete graph K_n $dn_{\square f}(G) = dn^{+}_{\square f}(G) = 2$. Also the inequality in Theorem 2.4 can be strict. For the graph given in Figure 2.1, $dn_{\square f}(G) < dn^{+}_{\square f}(G)$.

In the following theorems we give a minimal square free detour set of certain graphs.

Theorem 2.6 Let $G = (V, E)$ be a complete graph K_n $(n \geq 2)$. Then a set $S \subseteq V$ is a minimal square free detour set of G if and only if S consists of any two adjacent vertices of G.

Proof. Let $G = K_n$ be a complete graph of order $n(n \geq 2)$ and $V(G) = \{x_1, x_2, x_3, \dots, x_n\}$. Let $S = \{x_1, x_2\}$ be a set of two vertices of G. Let $x_i \in V$.

Case (i): Let $x_i \notin$ S . Then x_i $(3 \leq i \leq n)$ lies on a square free detour $x_1\, x_i\, x_2$ of length 2.

Case (ii): Let $x_i \in$ S and say $x_i = x_1$. Then x_i lies on a square free detour $x_i\, x_j\, x_2$ of length 2 where $x_j \notin$ S. Thus every vertex x_i of V lies on a square free detour in G and so S is a square free detour set of G. Since $|S| = 2$, S is a square free detour set of G with maximum number of vertices. Hence S is a minimal square free detour set of G.

Conversely, let S be a square free detour basis of G. Let S' be any set consisting of two adjacent vertices of G. Then as in the first part of this theorem, S' is a minimal square free detour set of G. If $|S'| \geq 3$, then S' as well as S cannot be the minimal square free detour sets. This leads to contradiction. Hence $|S| = |S'| = 2$ and it follows that S consists of any two adjacent vertices of G.

Theorem 2.7 Let $G = (V, E)$ be a complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$ with partitions $|X| = m$ and $|Y| = n$. Then a set of $S \subseteq V$ is a minimal square free detour set of G if and only if $S = Y$.

Proof. Let $G = K_{m,n}$ $(2 \leq m \leq n)$ be a complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$ with bipartite sets X and Y. Let $X = \{x_1, x_2, x_3, \dots, x_m\}$ and $Y = \{y_1, y_2, y_3, \dots, y_n\}$. Let $S = Y$ and $v \in V$. Then every v lies on the square free detour $y_i\, v\, y_j$ of length 2 for any distinct i and $j (1 \leq i, j \leq n)$. Thus $|S| = n$ and S is a square free detour set of G. Now we claim that S is minimal. If S^* is any subset of S, then by Theorem 1.3, S^* is not a square free detour set of G. Hence S is a minimal square free detour set of G.

Conversely, let S be a minimal square free detour set of G. Now assume that $|S| \geq n$. Let S' be any subset of S. Then by Theorem 1.3, S can not have elements from both the sets X and Y and hence any subset $S'$of S contains n vertices of Y. Then as in the first part of this theorem S' is a square free detour set of G with $|S'| = n$ and so S is not a minimal square free detour set of G, which is a contradiction. Thus S contains n vertices of Y.

Theorem 2.8 Let $G = (V, E)$ be an odd cycle C_n of order $n \geq 5$. Then a set $S \subseteq V$ is a minimal square free detour set of G if and only if S consists of any two adjacent vertices or three independent vertices of G.

Proof. Let $G = C_n$: $x_1\, x_2\, x_3 \ldots .\, x_n\, x_1$ be an odd cycle of order $n \geq 5$. Let $S = \{x_i, x_j : 1 \leq i, j \leq n\}$ be a set of two adjacent vertices of G. Then by Theorem 1.4, S is a square free detour set of G. Let $S = \{x_i, x_{j,}\, x_k : 1 \leq i, j, k \leq n\}$ be an independent set. If x_i lies on an $x_j - x_k$ square free detour, then the vertices on the $x_j - x_k$ geodesic lie either on $x_i - x_j$ square free detour or on $x_i - x_k$ square free detour. Similarly, if x_i lies on the $x_j - x_k$ geodesic, then the vertices on the $x_j - x_k$ geodesic lie either on $x_i - x_j$ square free detour or on $x_i - x_k$ square free detour. Thus, every vertex of V lies on a square free detour in G and so S is a square free detour set of G. Now we claim that S is minimal. If S^* is any subset of S with two vertices, then by Theorem 1.4, S^* is not a square free detour set of $G$and so S is a minimal square free detour set of G.

Conversely, assume that S is a minimal square free detour set of G. If $|S| = 2$, then by Theorem 1.4, S consists of any two adjacent vertices of G. If $|S| = 3$, then by Theorem 1.4, the vertices of S are independent. Now let $|S| \geq 4$, then S must be an independent set or contains a pair of adjacent vertices of G. In either case S is not a minimal square free detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices or three independent vertices of G.

Theorem 2.9 Let $G = (V, E)$ be an even cycle C_n of order $n \geq 6$. Then a set $S \subseteq V$ is a minimal square free detour set of G if and only if S consists of any two adjacent vertices or two antipodal vertices or three independent non-antipodal vertices of G.

Proof. Let $G = C_n$: $x_1\, x_2\, x_3 \ldots .\, x_n\, x_1$ be an even cycle of order $n \geq 6$. Let $S = \{x_i, x_j : 1 \leq i, j \leq n\}$ be a set of two adjacent vertices or two antipodal vertices of G. Then by Theorem 1.5, S is a square free detour set of G. Let $S = \{x_i, x_{j,}\, x_k : 1 \leq i, j, k \leq n\}$ be an independent set of three non-antipodal vertices of G. If x_i lies on an $x_j - x_k$ square free detour, then the vertices on the $x_j - x_k$ geodesic lie either on $x_i - x_j$ square free detour or on $x_i - x_k$ square free detour. Similarly, if x_i lies on the $x_j - x_k$ geodesic, then the vertices on the $x_j - x_k$ geodesic lie either on $x_i - x_j$ square free detour or on $x_i - x_k$ square free detour. Thus, every vertex of V lies on a square free detour in G and so S is a square free detour set of G. Now we claim that S is minimal. If S^* is any subset of S with two vertices, then by Theorem 1.5, S^* is not a square free detour set of G and so S is a minimal square free detour set of G.

Conversely, assume that S is a minimal square free detour set of G. If $|S| = 2$, then by Theorem 1.5, S consists of any two adjacent vertices or two antipodal vertices of G. If $|S| = 3$, then by Theorem 1.5, the set S is an independent set of non-antipodal vertices of G. Now let $|S| \geq 4$, then S must be an independent set of non-antipodal vertices or contains a pair of adjacent vertices or antipodal vertices of G. In either case S is not a minimal square free detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices or two antipodal vertices or three independent non-antipodal vertices of G.

Theorem 2.10 Let $G = (V, E)$ be an even cycle C_n of order $n = 4$. Then a set $S \subseteq V$ is a minimal square free detour set of G if and only if S consists of any two antipodal vertices of G.

Proof. The proof follows from Theorem 2.9.

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Corollary 2.11 Let $G = (V, E)$ be a connected graph.

(a) If G is the tree T with k end-vertices, then $dn^{+}_{\square f}(G) = k$.

(b) If G is the complete graph K_n, then $dn^{+}_{\square f}(G) = 2$.

(c) If G is the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$, then $dn^{+}_{\square f}(G) = n$.

(d) If G is the cycle C_n($n \geq 3$), then $dn^{+}_{\square f}(G) = \begin{cases} 2, & 3 \leq n \leq 5 \\ 3, & n \geq 6 \end{cases}$

Proof. (a) This follows from Theorem 1.2.

(b) This follows from Theorem 2.6.

(c) This follows from Theorem 2.7.

(d) This follows from Theorems 2,8, 2.9 and 2.10.

3 Conclusion

In this article we defined the minimal square free detour set and determined the upper square free detour number of standard graphs. The relationship between the square free detour number and the upper square free detour number has been established. We can extend this study to other graph classes, such as planar or bipartite graphs. Developing efficient algorithms for computing these numbers could also have significant implications. Additionally, exploring connections with other graph parameters, like domination or independence numbers, could lead to a deeper understanding of graph structure. Generalizations to weighted or directed graphs could further expand our understanding. Pursuing these directions could advance our knowledge of upper square-free detour numbers and their role in graph theory and network optimization.

4 Declarations

4.1 Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

4.2 Acknowledgements

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4.3 Study Limitations

This study focuses exclusively on standard graph classes, including trees, cycles, and complete bipartite graphs, to determine the upper square-free detour number. The analysis does not account for broader graph classes. The proposed methods have not been tested for large-scale networks or graphs with complex structures.

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