

(2+1) D Massless Dirac Equation in Anti de-Sitter Space

Lakhdar Sek^{1*}, Mokhtar Falek¹, Mustafa Moumni^{1,2}

¹Laboratory of Photonic Physics and Nano-Materials (LPPNNM), Department of Matter Sciences,
University of Biskra, ALGERIA

²Laboratory of Radiation and their Interactions with Matter (PRIMALAB), Department of Physics,
University of Batna1, ALGERIA

*Corresponding author

Abstract

In this study, we investigate the study of 2D graphene in the presence of external magnetic field in Anti de-Sitter space. Where the energy eigenvalues and the corresponding wave functions are obtained using Nikiforov-Uvarov method after submission this latter in the extended uncertainty principle of Heisenberg in order to show the curvature of the space. Moreover, the upper bound of the minimal uncertainty in momentum experimentally has calculate.

Keywords: Graphene, Anti de-Sitter space, Extended Uncertainty principle.

Introduction

Carbon in its allotropic forms such as graphite and diamond occupies a prominent place in various branches of science. In particular, one can imagine that graphite consists of the accumulation of thick layers of carbon in an atom, the so-called graphene. Theoretical scientific community, since experimental observations revealed the existence of electrical charge carriers that behave like massless Dirac quasi-particles [1-4]. The reason for this lies in the unusual molecular structure of graphene. Carbon atoms are arranged in a hexagonal lattice, similar to a honeycomb structure [5]. It has been observed that low-energy electronic excitations at the corners of the Brillouin zone of graphene can be described by (2 + 1) Dirac fermions with a linear scattering ratio (massless) [3, 4]. This effect offers the opportunity to test various aspects of relativistic phenomena, which generally require high energy, in experiments in condensed matter physics, such as: the chiral tunnel and the Klein paradox [6, 7]

On the other hand, among the various attempts to integrate gravity into the quantum world, there is one area that has generated great interest; it is the quantum theory of fields in curved space through generalizations of Heisenberg algebra, such as the extended uncertainty principle (EUP). The purpose of this extended principle is to account for the quantum fluctuations of the gravitational field in order to include gravity in the quantum mechanics domain. One of the consequences of this standardization is the existence of a minimal length scale of the Planck order [8]. We can combine this minimum length with a modification of Heisenberg's standard algebra by adding small corrections to the canonical commutation relations and thus changing their standard algebra; we quote here the work of Mignemi [8], who showed that Heisenberg relations in the (anti-) deSitter space are modified by adding corrections that are proportional to the cosmological constant. These modifications were also motivated by Doubly Special Relativity (DSR) [9, 10], string theory [11], non-commutative geometry



[12] and black hole physics [13, 14]. Effects of Newtonian gravity in quantum systems [15] and that the modification of inertia, which is predicted by some alternative theories of gravity on cosmic scales, can be derived naturally within the framework of the EUP [16].

In recent years, a large part of the research work has been devoted to the study of relativistic quantum mechanics with the EUP [17-19]. Some problems have also been solved in non-relativistic quantum mechanics; this happened despite the fact that we cannot derive any non-relativistic Schrödinger-like covariant equations from the Klein-Fock-Gordon covariant equation in the traditional field theoretical method of the deSitter (dS) models and anti-deSitter (AdS) [20-22].

In this work, we are interested in phenomenological models of quantum gravity. We study analytically in 2D spaces, the massless dirac equation in the position space representation for deformed quantum mechanics with EUP in an interaction with an external uniform magnetic field for this system.

The paper is organized as follows: In Sec.2, we provide an analysis of the AdS model while in Sec.3; we introduce the Nikiforov–Uvarov (NU) method used to solve the equation of our system. We expose in Sec.4 the explicit calculations of both eigenfunctions and eigenvalues of the deformed 2D graphene in a uniform magnetic field with AdS algebra. To sum up with a conclusion in Sec.5.

Review of the deformed quantum mechanics relation

The deformed Heisenberg algebra, which leads to the EUP model in AdS, is defined in 3D spaces by the following commutation relations [23, 24]:

$$[\mathbf{X}_i, \mathbf{X}_j] = \mathbf{0}; [\mathbf{P}_i, \mathbf{P}_j] = -i\hbar\lambda\epsilon_{ijk} \mathbf{L}_k; [\mathbf{X}_i, \mathbf{P}_j] = i\hbar(\delta_{ij} - \lambda\mathbf{X}_i\mathbf{X}_j) \quad (1)$$

where λ is a small positive deformation parameter. In the sense of quantum gravity, this parameter λ is calculated as the fundamental constant associated with the scaling factor of the expanding universe and is proportional to the cosmological constant $\Gamma = -3\lambda = -3a - 2$ where a is the radius of AdS [25]. L_k are the usual components of angular momentum and are expressed as follows:

$$\mathbf{L}_k = \epsilon_{ijk}\mathbf{X}_i\mathbf{P}_j \quad (2)$$

These components follow the usual momentum algebra;

$$[\mathbf{L}_i, \mathbf{P}_j] = i\hbar\epsilon_{ijk}\mathbf{P}_k; \quad (3)$$

$$[\mathbf{L}_i, \mathbf{X}_j] = i\hbar\epsilon_{ijk}\mathbf{X}_k;$$

$$[\mathbf{L}_i, \mathbf{L}_j] = i\hbar\epsilon_{ijk}\mathbf{L}_k$$

The AdS deformed algebra (1) gives rise to modified Heisenberg uncertainty relations:

$$\Delta\mathbf{X}_i\Delta\mathbf{P}_i \geq \frac{\hbar}{2}(1 + \lambda(\Delta\mathbf{X}_i)^2) \quad (4)$$

Where we have chosen the states for which $\langle X_i \rangle = 0$.

It also generates a minimum uncertainty in momentum. For simplicity, if we assume isotropic uncertainties $X_i = X$, we get:

$$(\Delta\mathbf{P}_i)_{\min} = \hbar\sqrt{\lambda} \quad (5)$$

So the noncommutative operators X_i and P_i satisfy the AdS algebra (1) with the rescaled uncertainty relations in position space (4). In what follows, we represent these operators as functions of the usual x_i and p_i operators fulfilling the ordinary canonical commutation relations in position space; this is done with the following transformations:

$$\mathbf{X}_i = \frac{\mathbf{x}_i}{\sqrt{1 - \lambda r^2}} \quad (6)$$

$$\mathbf{P}_i = -i\hbar\sqrt{1 - \lambda r^2}\partial\mathbf{x}_i \quad (7)$$

Here the variable r vary in the domain $\left] -\frac{1}{\sqrt{\lambda}}, \frac{1}{\sqrt{\lambda}} \right[$.

Nikiforov-Uvarov method

NikiforovUvarov's (NU) approach is mainly based on the hypergeometric differential equation. The formulas used in the NU method reduce the differential equations of the second order with a suitable coordinate transformation to the hypergeometric type (note that $s \equiv s(x)$ and the prime numbers denote the derivatives):

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \quad (8)$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are at most polynomials of the second degree [26, 27], while the degree of the polynomial $\tilde{\tau}(s)$ is strictly less than

$$\psi(s) = \phi(s)y(s) \quad (9)$$

Eq. (8) becomes [27]:

$$\sigma(s)y''(s) + \tau(s)y'(s) + \Lambda y(s) = 0 \quad (10)$$

Where

$$\pi(s) = \sigma(s) \frac{d}{ds}(\ln\phi(s)) \text{ and } \tau(s) = \tilde{\tau}(s) + 2\pi(s) \quad (11)$$

Moreover, Λ defined by;

$$\Lambda_n + n\tau' + \frac{n(n+1)}{2}\sigma'' = 0 \text{ and } n = 0, 1, 2, \dots \quad (12)$$

The energy eigenvalues of the system are determined from the previous equation, to find them we need to evaluate $\pi(s)$ and identify them first:

$$k = \Lambda - \pi'(s) \quad (13)$$

We get the solution of the quadratic equation for $\pi(s)$, which is a polynomial of s :

$$\pi(s) = \left(\frac{\sigma' - \tilde{\tau}}{2}\right) \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + \sigma k} \quad (14)$$

It should be noted that in computing $\pi(s)$ the determination of k is critical point and it is reached by indicating that the expression under the square root in (14) must be a square polynomial; this gives us a general quadratic equation for k . We use (11) and the Rodrigues relation to evaluate the polynomial solutions $y_n(s)$:

$$y_n(s) = \frac{C_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)] \quad (15)$$

Where C_n are constants that used to normalize the solutions, the weighted function satisfy the following relationship:

$$\frac{d}{ds} [\sigma(s)\rho(s)] = \tau(s)\rho(s) \quad (16)$$

This last equation relates to the classical orthogonal polynomials and we write the orthogonality relations for the polynomial solutions as follows:

$$\int_a^b y_n(s)y_m(s)\rho(s)ds = 0 \text{ if } m \neq n \quad (17)$$

2D Massless Dirac Equation in Anti de-Sitter Space

The electron in quantum theory of graphene is a massless fermion that moves at a velocity $V_F = (1.12 \pm 0.02) \times 10^6 \text{ ms}^{-1}$, called the Fermi velocity verify the relativistic massless Dirac equation. The discovery of graphs gives us the opportunity to test various effects of QED, such as the “small paradox”, since this effect cannot be observed in particle physics [28]. In this section, we are interested in solving the Dirac equation without dimensional mass (1 + 2) in the presence of a constant external magnetic field $\vec{A} = \frac{B}{2} (-y, x, 0)$. The Hamiltonian of the massless (2+1)-dimensional Dirac equation is [29];

$$(\hat{\alpha} \cdot \mathbf{p})\Psi(\mathbf{r}) = \frac{E}{V_F}\Psi(\mathbf{r}) \quad (18)$$

Where $\hat{\alpha}$ is the usual Dirac matrices and we may assume that the four-component spinor Ψ is of the form $\Psi(\mathbf{r}) = (\Psi_a(\mathbf{r}), \Psi_b(\mathbf{r}))$.

We use the AdS algebra definition (eqs.6 and 7) and involving the spinor above, to rewrite this equation in the deformed momentum space:

$$\hat{\sigma} \cdot (\mathbf{p} - e\mathbf{A})\Psi_b(\mathbf{r}) = \frac{E}{V_F}\Psi_a(\mathbf{r}) \quad (19-a)$$

$$\hat{\sigma} \cdot (\mathbf{p} - e\mathbf{A})\Psi_a(\mathbf{r}) = \frac{E}{V_F}\Psi_b(\mathbf{r}) \quad (19-b)$$

With $\hat{\sigma}$ designates the Pauli matrices. Then, we eliminate $\Psi_b(\mathbf{r})$ in favor of $\Psi_a(\mathbf{r})$, to obtain the following equation:

$$[\hat{\sigma} \cdot (\mathbf{p} - e\mathbf{A})]^2\Psi_a(\mathbf{r}) = \frac{E^2}{V_F^2}\Psi_a(\mathbf{r}) \quad (20)$$

According to the following relations:

$$(\hat{\sigma} \cdot \mathbf{A})(\hat{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\hat{\sigma} \cdot (\mathbf{A} \times \mathbf{B}) \quad (21)$$

The eq (20) becomes:

$$(\mathbf{p}^- \cdot \mathbf{p}^- + i\hat{\sigma} \cdot (\mathbf{p}^- \times \mathbf{p}^-))\Psi_a(\mathbf{r}) = \frac{E^2}{V_F^2}\Psi_a(\mathbf{r}) \quad (22)$$

Where $\mathbf{p}^- = \left(\sqrt{1 - \lambda r^2} \mathbf{p} - \left(\frac{eB}{2} \right) \left(\frac{-y}{\sqrt{1 - \lambda r^2}} \vec{i} + \frac{x}{\sqrt{1 - \lambda r^2}} \vec{j} \right) \right)$.

After a straightforward calculation of eq (22), we obtain:

$$\left[(1 - \lambda r^2) \mathbf{p}^2 + \alpha \frac{r^2}{1 - \lambda r^2} + i\hbar \lambda \mathbf{r} \cdot \mathbf{p} - \gamma L_z - eB\hbar\sigma_z - \frac{E^2}{V_F^2} \right] \Psi_a(\mathbf{r}) = 0 \quad (23)$$

Here the parameters;

$$\alpha = \frac{e^2 B^2}{4} - \frac{eB}{2} \lambda \hbar \sigma_z \quad (24)$$

$$\gamma = eB + \lambda \hbar \sigma_z \quad (25)$$

To solve the eq.(23), we introduce the polar coordinates in position space (r, ϕ) , and we use the following ansatz $\Psi_a(\mathbf{r}) = \exp(im_l\phi) R_{n,l}(r) \chi_\tau$, where n is the radial quantum number, m_l and $\tau = \pm 1$ are, respectively, the eigenvalues of angular momentum and spin operators, and $\chi_{+1}^T = (1, 0)$, $\chi_{-1}^T = (0, 1)$ are the spin functions; to obtain

$$\left[(1 - \lambda r^2) \left(\frac{d}{dr} \right)^2 - \frac{m_l^2 (1 - \lambda r^2)}{r^2} - \frac{\eta r^2}{\hbar^2 (1 - \lambda r^2)} + \varepsilon \right] R_{n,l}(r) = 0 \quad (26)$$

With

$$\eta = \frac{\hbar^2}{4l_B^4} - \frac{\lambda\hbar^2}{2l_B^2}\tau \quad (27)$$

$$\varepsilon = \frac{E^2}{\hbar^2 v_F^2} + \frac{\tau}{l_B^2} + m_l \left(\frac{1}{l_B^2} + \lambda\tau \right) \quad (28)$$

Where $l_B = \sqrt{\frac{\hbar}{eB}}$ is the fundamental length scale in the presence of a magnetic field.

We consider the transformations and the proceeding steps to follow by using the Nikiforov-Uvarov (NU) in order to obtain the energy spectrum and the corresponding wave function as in Ref [30].

Until we get the energy spectrum in the form:

$$E_{n,m_l,\tau}^\lambda = \pm \frac{\hbar v_F}{l_B} [(2n + m_l + 1) \sqrt{(1 - 2\lambda l_B^2 \tau + \lambda^2 l_B^4) + \lambda l_B^2 (4n(n + m_l + 1) + (2 - \tau)m_l + 1) - (m_l + \tau)}]^{1/2} \quad (29)$$

We notice that the energy spectrum of our system has n^2 dependence of the energy levels, which corresponds to a confinement at the high-energy area; our result is equivalent to the energy of a spinless relativistic quantum particle in a square well potential.

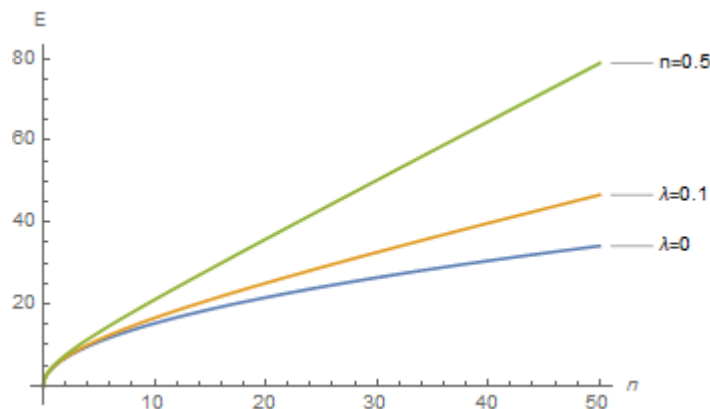


Figure.1: Energy spectrum as a function of n for the different values of the parameter λ

In **Fig. 1**, We represent the energy level as a function of the quantum number n for different values of λ , we have chosen $\lambda = 0, 0.1, 0.5$ and $m_l = 0, s = 1/2$. As a result, we observe that for a fixed value of n , the energy E increases monotonically with increasing parameter EUP. The effect of the EUP parameter on the energy levels can be observed λ^2 , where $\lambda = 0$ corresponds to the case of normal quantum mechanics.

Now, let us conclude the corresponding wave function after employing and following the steps as NU method to get that form;

$$\Psi_n(r, \phi) = C_n 2^{\frac{m_l}{2}} \exp(i m_l \phi) (1 - \lambda r^2)^{\mu/2} (\lambda r^2)^{\frac{m_l}{2}} P_n^{(m_l, \mu-1/2)}(1 - 2\lambda r^2) \quad (30)$$

Where C_n is the normalization constant.

We can test this finding in many ways such as; if we put $\lambda = 0$, we obtain the ordinary expression of the spectrum in the usual space [31]

$$\mathbf{E}_{n,0,1}^0 = \pm \frac{\hbar V_F}{l_B} \sqrt{2n} \quad (31)$$

In order to get the upper limit of the deformation parameter λ and to output it up to the first order, we use the s-states of the energy eigenvalues

$$\mathbf{E}_{n,0,1}^\lambda = \mathbf{E}_{n,0}^{\lambda=0} + \mathbf{E}_{n,0}^{\lambda=0} \left(\frac{l_B^2 (2n+1)}{2\sqrt{2n}} \right) \lambda \quad (32)$$

$$\mathbf{E}_{n,0,1}^\lambda = \pm \frac{\hbar V_F}{l_B} [2n(1 + \lambda l_B^2 (2n+1))]^{1/2} \quad (33)$$

Where $\mathbf{E}_{n,0}^{\lambda=0}$ defined in eq (31).

The deviation from the usual case caused by the modified commutation relations (1) of the nth energy level is provided by:

$$\Delta \mathbf{E}_{n,0,+1}^\lambda = \mathbf{E}_{n,0}^{\lambda=0} \left(\frac{l_B^2 (2n+1)}{2\sqrt{2n}} \right) \lambda \quad (34)$$

We use the experimental results of the relativistic Landau levels in graphene [32]. In the absence of deformation, the energy for the Landau levels $n = 1$, for a magnetic field of strength $B = 18\text{T}$ is $\mathbf{E}_{n,0} = (172 \pm 3)\text{meV}$. At this point, if we put $n = 1$ in the s-states of the energy eigenvalues (31) and considering that the uncertainty in the energy is $\Delta \mathbf{E}_{n,0,+1}^\lambda < 6\text{meV}$, we get the following constraint:

$$\lambda < 25 \times 10^6 \text{m}^{-2} \quad (35)$$

This leads to the following upper bound of the minimal uncertainty in momentum $\Delta P_{\min} = \hbar \sqrt{\lambda} < 2.64 \times 10^{-27} \text{kg m s}^{-1}$.

Conclusion

In this paper, we investigated the exact solutions of the 2D massless Dirac equation with a uniform external magnetic field in the context of deformed quantum mechanics with anti-deSitter commutation relationships. These AdS deformations lead to a minimal non-zero uncertainty in the measurement of the momentum. The NikiforovUvarov method was used and thus we get the analytical expressions of the bound state energies and the wave functions of the system, in this case, we express the functions analytically in terms of the system in relation on Jacobi's polynomials and the corresponding eigenenergies with additional corrections depending on the deformation parameter λ . Our results show that the deformed spectrum remains discrete even with large values of the principal quantum number and thus the EUP deformation eliminates the degeneracy of the spectrum (no deformations) found in the normal case. We have thus obtained an experimental parameter of the limit deformation. Finally, a satisfactory upper limit for the EUP was determined in order to see the effect of the deformation in physical systems and to compare it with the experimental results of the relativistic Landau levels in graphene.

References

- [1] K.S. Novoselov, et al. Science 306 (2004) 666.
- [2] K.S. Novoselov, et al. Proc. Natl. Acad. Sci. 102 (2005) 10451.
- [3] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, A.K. Geim, Rev. Modern Phys. 81 (2009) 109.

- [4] N. Peres, *Rev. Modern Phys.* 82 (2010) 2673.
- [5] Charles L. Fefferman, Michael I. Weinstein, *J. Amer. Math. Soc.* 25 (2012) 1169.
- [6] M.I. Katsnelson, K.S. Novoselov, A.K. Geim, *Nat. Phys.* 2 (2006) 620.
- [7] M.I. Katsnelson, *Mater. Today* 10 (2007) 20.
- [8] S. Mignemi. *Mod.Phys. Lett. A* 25, 1697 (2010) 17
- [9] G. Amelino-Camelia. *Phys. Lett. B* 510,255 (2001)
- [10] G. Amelino-Camelia. *Int. J. Mod. Phys. D* 11, 35 (2002)
- [11] S. Capozziello, G. Lambiase, G. Scarpetta. *Int. J. Theor. Phys.* 39, 15 (2000)
- [12] M.R. Douglas and N.A. Nekrasov. *Rev. Mod. Phys.* 73, 977 (2001)
- [13] F. Scardigli. *Phys. Lett. B* 452, 39 (1999)
- [14] F. Scardigli and R. Casadio. *Class. Quant. Grav.* 20, 3915 (2003)
- [15] V.E. Kuzmichev and V.V. Kuzmichev. *Eur. Phys. J. C* 80, 248 (2020)
- [16] J. Gin' e and G. G. Luciano. *Eur. Phys. J. C* 80, 1039 (2020)
- [17] B. Hamil and M. Merad. *Eur. Phys. J. Plus* 133, 174 (2018)
- [18] W.S. Chung, H. Hassanabadi and N. Farahani. *Mod. Phys. Lett. A* 34, 1950204 (2019)
- [19] M. Hadj Moussa and M. Merad. *Few-Body Syst.* 59, 44 (2018)
- [20] B. Hamil, M. Merad and T. Birkandan. *Eur. Phys. J. Plus* 134, 278 (2019)
- [21] S. Ghosh and S. Mignemi. *Int. J. Theor. Phys.* 50,1803 (2011)
- [22] M. Falek, N. Belghar and M. Moumni. *Eur. Phys. J. Plus.* 135, 335 (2020)
- [23] S. Mignemi. *Class.Quant. Grav.* 29, 215019 (2012)
- [24] M.M. Stetsko. *J. Math. Phys.* 56, 012101(2015)
- [25] B. Bolen, M. Cavagli`a. *Gen. Relativ. Gravit.* 37, 1255–1262 (2005)
- [26] H. Egrifes, D. Demirhan, F. Buyukkili,c. *Phys. Scripta* 59, 195–198 (1999)
- [27] A.F. Nikiforov, V.B. Uvarov, *Special Functions of Mathematical Physics* (Birkhauser, Basel, 1988).
- [28] A.D. Guclu, P. Potasz, M. Korkusinski, P. Hawrylak, *Graphene Quantum Dots* (Springer, Berlin, 2014)
- [29] Hamil, B., & Merad, M. *Few-Body Systems*, 60(2), 36. (2019)
- [30] Sek, Lakhdar, Mokhtar Falek, and Mustafa Moumni. *Int. J. Mod.Phys.A.*2150113 (2021)
- [31]V. Santos, R.V. Maluf, C.A.S. Almeida, *Ann. Phys.* 349, 402 (2014)
- [32] Z. Jiang et al. *Phys. Rev. Lett.* 98, 197403 (2007)