Study of Nonlinear Time Fractional Generalized Burger Equation with Proportional Delay via q-HAM

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ABSTRACT

The q-homotopy analysis method has been successfully implemented to study the nonlinear time-fractional generalized Burger equation with proportional delay (in brief, $_{TF}\text{GBE}^{PD}$), considering fractional derivative of Caputo type. The computed results are analyzed in terms of absolute error (and relative error norms in the fourth order approximation) for two test problems of nonlinear $_{TF}\text{GBE}^{PD}$. The relative error is reported for $\alpha = 0.8, 0.9, 1$ for different values of $x, t \in [0, 1]$. To guarantee the convergence of the results, $\hbar$-curves are depicted graphically for different values of the fractional order (α). The findings shows that the proposed results are much better than the existing results and approaches towards the exact solution.

Keywords: Fractional integral operator; Caputo derivative; time-fractional generalized Burger equation with proportional delay; q-homotopy analysis method, convergence-control parameters.

1 INTRODUCTION

Time delay differential equation is characterized as a differential equation where the unknown function is given at a specific time while the value of the function depend on the previous time-state simultaneously. In general for a dynamical model, we assume the hypothesis that the system depend only at the present time-state, which need not to be feasible for each physical phenomena. This is why it is quite worth to examine and analyze the delayed system via some effective techniques [6]. Moreover, time-delay involves in various realistic models occurred in economics, biology, physics, chemistry, engineering and medicine. Delay differential equation perform a significant role in vigorous physical/biological models for both integer order and fractional order system, see [7, 8, 9, 15].

Burger's equation is a generalisation of Navier-Stokes equation,represents an elementary nonlinear system for turbulence flow.Delayed Burger equation plays an important role in kinematic wave theory of traffic flow,reaction-diffusion and convection-diffusion systems. In general, the study of such delayed equations is a very tough experience.Nevertheless, numerous vigorous effective schemes have been introduced to study such types of the differential equations with delay, for instance, predictor-corrector algorithm [15], pseudospectral scheme [10], spectral collocation schemes [11], reduce differential transform method [12, 13], Adams-Bashforth-Moulton algorithm [14], and a scheme based on the definition of GL [16].The nonlinear fractional partial differential equations with proportional delay have been studied via Extended FRDTM [21] (and extended DTM [12] for $\alpha = 1$ only), HPM [17], FVIM[18], HPTM [19] and Natural transform decomposition method [20]. Tanthanuch [22] implemented group analysis method for nonhomogeneous inviscid Burgers equation with delay.

Liao [3]has suggested homotopy analysis method (HAM), which is very rigorous scheme to study several kind of nonlinear equations, e.g.,differential-integral, algebraic equations, PDEs, ordinary differential equations as well as coupled systems and fractional models of these equations.Different from all perturbation and non perturbation methods for nonlinear problems HAM produce an effective and easy technique to assure the convergent solutions by choosing

different base functions[4], see [1, 2, 5, 28, 29] for more on q-HAM. Best of my knowledge, the nonlinear $_{TF}\text{GBE}^{PD}$ has not been studied via q-HAM, therefore we have adopted q-HAM to study the following nonlinear $_{TF}\text{GBE}^{PD}$ (1):

$$\mathcal{N}[\mathcal{D}_t^\alpha \phi] = \mathcal{D}_t^\alpha \phi(x,t) - \mathcal{F}\left(x, \phi(r_0 x, s_0 t), \phi_x(r_1 x, s_1 t), .., \phi_x^{(m)}(r_n x, s_n t)\right) = 0 \tag{1}$$

For $0 < \alpha \le 1$, with the initial condition: $\phi(x,0) = f(x), x \in [0,1]$, where $\mathcal{N}$ is the nonlinear operator, f is a smooth function, $\phi(x,t)$ is the unknown to be computed, and $\mathcal{D}_t^\alpha \phi(x,t)$ is the Caputo derivative $\phi \in \mathbf{C}_\mu$ ($\mu \ge -1$) of order α:

$$\mathcal{D}_t^\alpha \phi(t) := \mathcal{J}_t^{m-\alpha}\left(\frac{d^m \phi}{dt^m}\right), \quad m-1 < \alpha \le m \in \mathbf{N}, \tag{2}$$

where $\mathcal{J}_t^\alpha \phi(t)$ ($0 < \alpha \le 1$) is Riemann-Liouville integration of $\phi \in \mathbf{C}_\mu$ which is defined by $\mathcal{J}_t^0 \phi(t) = \phi(t)$, $\mathcal{J}_t^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-\tau)^{\alpha-1}\phi(\tau)\,d\tau, t_0 \ge 0$ when $\alpha > 0$,

It is easy to see that $\mathcal{J}_t^\alpha (t - t_0)^\gamma = \frac{\Gamma(1+\gamma)(t-t_0)^{\gamma+\alpha}}{\Gamma(1+\gamma+\alpha)}$, for $t \ge t_0$; function $\phi : \mathbf{R}^+ \to \mathbf{R}$ is said to be in class $\mathbf{C}_\mu, \mu \in \mathbf{R}$ if $\exists \ell \in \mathbf{R}$ ($\ell > \mu$) and $h \in C[0, \infty)$ with property that $\phi(t) = t^\ell h(t)$ for each $t \in \mathbf{R}^+$. In addition if $\phi^{(m)} \in \mathbf{C}_\mu, m \in \mathbf{N}$, then $\phi \in \mathbf{C}_\mu^m$ [23, 24]. If we set $m-1 < \alpha \le m \in \mathbf{N}, \phi \in \mathbf{C}_\mu^m, \mu \ge -1$ and $\gamma > \alpha - 1$, then Caputo operator $\mathcal{D}_t^\alpha$ have the following properties: $(a)\mathcal{D}_t^\alpha \mathcal{J}_t^\alpha \phi(t) = \phi(t)$; $(b)\mathcal{J}_t^\alpha \mathcal{D}_t^\alpha \phi(t) = \phi(t) - \sum_{k=0}^{m-1}\mathcal{D}_t^k \phi(t_0)\frac{(t-t_0)^k}{\Gamma(1+k)}, t_0 \ge 0$; $(c)\mathcal{D}_t^\alpha(t-t_0)^\gamma = \frac{\Gamma(1+\gamma)(t-t_0)^{\gamma-\alpha}}{\Gamma(1+\gamma-\alpha)}$. We refer interested readers to [23, 24, 25, 26, 27] and the papers therein for details on the properties of these operators.

2 DESCRIPTION OF OPTIMAL q-HOMOTOPY ANALYSIS METHOD

For Caputo time fractional PDEs: $\mathcal{N}[\mathcal{D}_t^\alpha \phi] = 0$, $m-1 < \alpha \le m$, the zeroth order deformation equation can be written as-

$$(1-q)\ell[\Psi(x,t;q) - \phi_0(x,t)] = q\hbar H(x,t)\mathcal{N}[\mathcal{D}_t^\alpha \Psi(x,t;q)], \quad q \in [0,1] \tag{3}$$

ℓ be an auxiliary linear operator with $\ell[C] = 0, C$ a constant, $0 \ne h \to$ auxiliary parameter while $0 \ne H(x,t) \to$ auxiliary function. Note that as q varies from 0 to 1, the solution $\Psi(x,t;q)$ varies from $\phi_0(x,t)$, the initial guess to $\phi(x,t)$, the exact solution of the original problem (Equation 3 yields $\Psi(x,t;0) = \phi_0(x,t)$ and $\Psi(x,t;1) = \phi(x,t)$).

As mentioned in [3], $\Psi(x,t;q)$ has its Taylor series expansion of form: $\Psi(x,t;q) = \phi_0(x,t) + \sum_{j=1}^\infty \phi_j(x,t)q^j$, where $\phi_j(x,t) = \frac{1}{j!}\frac{\partial^j \Psi(x,t;q)}{\partial q^j}\Big|_{q=0}$. Set $\ell, \hbar, \phi, H$ so as the computed results converges to $\phi(x,t)$ as $q = 1$, i.e.,
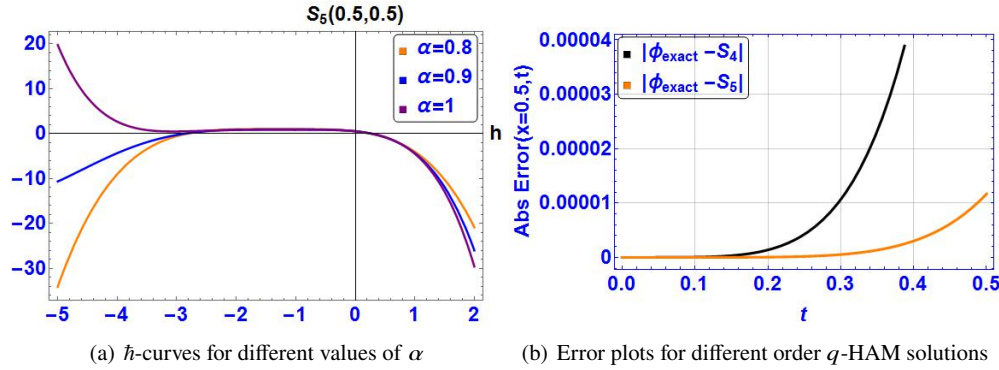
$$\phi(x,t) = \Psi(x,t;1) = \phi_0(x,t) + \sum_{j=1}^\infty \phi_j(x,t) \tag{4}$$

Write $\bar{\phi}_j = \{\phi_0, \phi_1, \ldots, \phi_j\}$, then j-th order deformation equation (the equation which is obtained by differentiating j-times equation 3 with respect to q, then substitute $q = 0$ and divide the equation by $j!$) is

$$\ell[\phi_j(x,t) - \chi_j \phi_{j-1}(x,t)] = \hbar H(x,t)\mathcal{R}_j[\phi_{j-1}(x,t)], \forall j \ge 1, \phi_0(x,t) = \sum_{r=0}^{m-1}\phi^{(r)}(x,0+)\frac{t^r}{r!}, \tag{5}$$

Where $\mathcal{R}_j[\phi_{j-1}(x,t)] = \frac{1}{(j-1)!}\frac{\partial^{j-1}(\mathcal{N}[\mathcal{D}_t^\alpha \Psi(x,t;q)])}{\partial q^{j-1}}\Big|_{q=0}$ and $\chi_j = 1$ if $j > 1$ and 0 otherwise. Operate ℓ^{-1} on both side of (5), we obtain

$$\phi_j(x,t) = \chi_j \phi_{j-1}(x,t) + \hbar \ell^{-1}\left\{H(x,t)\mathcal{R}_j[\phi_{j-1}(x,t)]\right\}, \quad j \ge 1. \tag{6}$$

(a) $\hbar$-curves for different values of α

(b) Error plots for different order q-HAM solutions

Fig. 1 (*a*) 2D plot of $\hbar$-curves for $\alpha = 0.8, 0.9, 1$ and $x = t = 0.5$ for S_m $(m = 5)$ solution; (*b*) 2D plots of absolute error for $x = 0.5, t \in [0, 1/2]$ in S_m $(m = 4, 5)$ solutions of $_{TF}$GBE^{PD} (7).

Equation (6) with the initial guess: $\phi_0(x, t) = \sum_{r=0}^{m-1} \phi^{(r)}(x, 0+)\frac{t^r}{r!}$ can be solved for $\phi_j, j \geq 1$. The acceptable interval of the controlling parametr $\hbar$ is the line segment approximately parallel to the horizontal axis, in $\hbar$-curve and so called $\hbar$-region. The optimal value of $\hbar$ is the values of $\hbar$ within the $\hbar$-region for which the computed error is minimum. The convergence of optimal q-HAM is guaranteed in [4, 5].

2.1 STUDY OF NONLINEAR $_{TF}$GBE^{PD} via q-HAM

Example 1 In the first example, consider initial valued nonlinear $_{TF}$GBE^{PD} of the form [17, 18, 19, 21]

$$\mathcal{D}_t^\alpha \phi(x, t) = \frac{\partial^2}{\partial x^2}[\phi(x, t)] + \frac{\partial}{\partial x}\left[\phi\left(x, \frac{t}{2}\right)\right]\phi\left(\frac{x}{2}, \frac{t}{2}\right) + \frac{1}{2}\phi(x, t), \phi(x, 0) = x, x, t \in [0, 1], 0 < \alpha \leq 1. \quad (7)$$

Here nonlinear operator:

$$\mathcal{N}[\Phi(x, t; q)] = \mathcal{D}_t^\alpha \Phi(x, t; q) - \frac{\partial^2}{\partial x^2}[\Phi(x, t; q)] - \frac{\partial}{\partial x}\left[\Phi\left(x, \frac{t}{2}; q\right)\right]\Phi\left(\frac{x}{2}, \frac{t}{2}; q\right) - \frac{1}{2}\Phi(x, t; q)$$

Zeroth order deformation equation is taken same as in Equation 3. If we take $H(x, t) = 1$, the auxiliary linear operator: $\ell[\Psi(x, t; q)] = \mathcal{D}_t^\alpha[\Psi(x, t; q)]$, then j-th order deformation equation for $_{TF}$GBE^{PD} (7) is as follows

$$\phi_j(x, t) = \chi_j \phi_{j-1}(x, t) + \hbar \mathcal{J}_t^\alpha\left[\mathcal{R}_j[\phi_{j-1}(x, t)]\right], j \geq 1. \quad (8)$$

Where
$\mathcal{R}_j[\phi_{j-1}(x, t)] = \mathcal{D}_t^\alpha \phi_{(j-1)}(x, t) -$
$\frac{\partial^2}{\partial x^2}\phi_{(j-1)}(x, t) - \sum_{i=0}^{j-1} \frac{\partial}{\partial x}\left[\phi_i\left(x, \frac{t}{2}\right)\right]\phi_{(j-1-i)}\left(\frac{x}{2}, \frac{t}{2}\right) - \frac{1}{2}\phi_{(j-1)}(x, t)$.
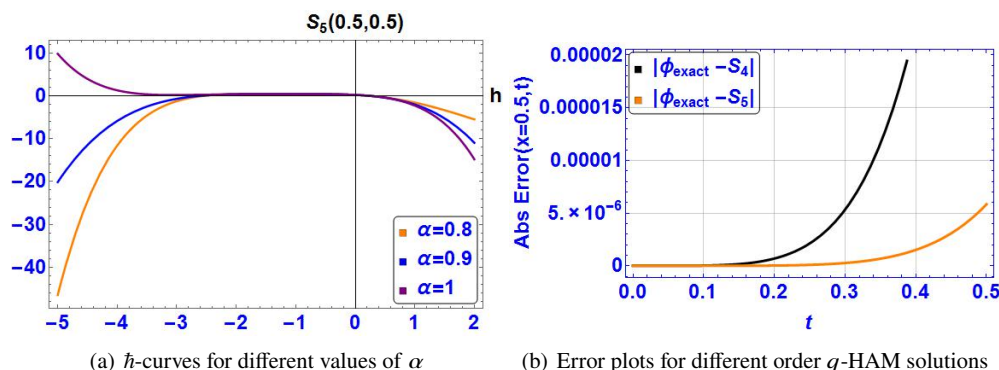On solving the recurrence relation (8) with the initial guess: $\phi_0(x, t) = \phi(x, 0) = x$, we get

$$\phi_1(x, t) = -\frac{\hbar x t^\alpha}{\Gamma(\alpha + 1)}; \quad \phi_2(x, t) = \frac{\sqrt{\pi}2^{-3\alpha-1}(2^\alpha + 2)\hbar^2 x t^{2\alpha}}{\Gamma(\alpha + 1)\Gamma\left(\alpha + \frac{1}{2}\right)} - \frac{\hbar x(\hbar + 1)t^\alpha}{\Gamma(\alpha + 1)}; \cdots \quad (9)$$

Continuing this process in similar manner, one can get the values of $\phi_j(x, t)$ for $j \geq 3$, fifth order approximate solution for the nonlinear $_{TF}$GBE^{PD} (7), in terms of controlling parameters $\hbar$ is as follows:

$$S_5(x, t, \hbar) = \phi_0(x, t) + \sum_{i=1}^{5} \phi_i(x, t). \quad (10)$$

Table 1 Comparison of absolute error (E_{abs}) in the solutions S_m ($m = 4, 5$) of $_{TF}\text{GBE}^{PD}$ (7) with the error in S_4 via HPM[17], and the relative error (RE) for $\alpha = 0.8, 0.9, 1.0$ with optimal values of $\hbar$.

Optimal $\hbar \rightarrow$		RE			E_{abs} ($\alpha = 1$)		
		-1.20055	-1.1396	-1.09836	-1.09736	-1.07419	
x	t	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	S_4	S_5	HPM [17]
0.25	0.25	5.8977E-05	1.1035E-05	2.122401E-06	4.8721E-07	4.6598E-08	2.1230E-06
	0.50	2.2944E-05	1.8045E-06	7.094268E-05	1.4370E-06	3.1030E-07	7.0943E-05
	0.75	6.4704E-04	2.4478E-04	5.634807E-04	4.7401E-05	2.8905E-06	5.6348E-04
	1.00	2.7443E-04	9.3715E-05	2.487124E-03	1.4524E-05	5.7745E-07	2.4871E-03
0.50	0.25	5.8977E-05	1.1035E-05	4.244802E-06	9.7442E-07	9.3195E-08	4.2450E-06
	0.50	2.2944E-05	1.8045E-06	1.418854E-04	2.8741E-06	6.2060E-07	1.4189E-04
	0.75	6.4704E-04	2.4478E-04	1.126961E-03	9.4803E-05	5.7809E-06	1.1270E-03
	1.00	2.7443E-04	9.3715E-05	4.974248E-03	2.9049E-05	1.1549E-06	4.9743E-03
0.75	0.25	5.8977E-05	1.1035E-05	6.369688E-06	1.4616E-06	1.3979E-07	6.3670E-06
	0.50	2.2944E-05	1.8045E-06	2.128250E-04	4.3111E-06	9.3089E-07	2.1283E-04
	0.75	6.4704E-04	2.4478E-04	1.690020E-03	1.4220E-04	8.6714E-06	1.6905E-03
	1.00	2.7443E-04	9.3715E-05	7.461370E-03	4.3573E-05	1.7324E-06	7.4614E-03

(a) $\hbar$-curves for different values of α

(b) Error plots for different order q-HAM solutions

Fig. 2 (*a*) 2D plot of $\hbar$-curves for $\alpha = 0.8, 0.9, 1$ and $x = t = 0.5$; for fifth order solutions (*b*) 2D plots of absolute error for $x = 0.5, t \in [0, 1/2]$; in S_m ($m = 4, 5$) solutions of $_{TF}\text{GBE}^{PD}$ (11).

When $\hbar = -1, \alpha = 1$ the solution (10) reduces to $S_5(x, t) = x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right)$, which is similar to the exact solution $\phi(x, t) = xe^t$. The same exact series solution is obtained via DTM, HPM, FRDTM, AVIM and HPTM [12, 17, 18, 19, 21].

Example 2 Consider initial valued nonlinear $_{TF}\text{GBE}^{PD}$ of the following form as described in [17, 18, 19, 21]

$$\mathcal{D}_t^\alpha \phi(x, t) = \frac{\partial^2}{\partial x^2}\left(\phi(x, \frac{t}{2})\right)\phi(x, \frac{t}{2}) - \phi(x, t), \phi(x, 0) = x^2, x, t \in [0, 1], 0 < \alpha \le 1. \tag{11}$$

Analogous to Example 1, the q-HAM yields the following j-th order deformation equation for Example 2

$$\phi_j(x, t) = \chi_j \phi_{j-1}(x, t) + \hbar \mathcal{J}_\alpha^t \left[\mathcal{R}_j[\phi_{j-1}(x, t)]\right], j \ge 1, \text{where} \tag{12}$$

$$\mathcal{R}_j[\phi_{j-1}(x, t)] = \mathcal{D}_t^\alpha \phi_{(j-1)}(x, t) - \sum_{i=0}^{j-1}\frac{\partial^2}{\partial x^2}\left[\phi_i(x, \frac{t}{2})\right]\phi_{(j-1-i)}(x, \frac{t}{2}) + \phi_{(j-1)}(x, t).$$

The recurrence relation (12) subject to the initial guess: $\phi_0(x, t) = x^2$, yields

Table 2 Comparison of absolute error (E_{abs}) in the solutions S_m ($m = 4, 5$) of $_{TF}$GBE^{PD} (11) with the error in S_4 via HPM[17], and the relative error (RE) for $\alpha = 0.8, 0.9, 1.0$ with optimal values of $\hbar$.

Optimal		RE			E_{abs} ($\alpha = 1$)		
$\hbar \rightarrow$		-1.19407	-1.12687	-1.09836	-1.09736	-1.07419	
x	t	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	S_4	S_5	HPM [17]
0.25	0.25	8.5638E-05	8.2546E-06	1.8044E-06	1.2180E-07	1.1649E-08	5.3000E-07
	0.50	7.9700E-04	1.6716E-04	3.0022E-06	3.5926E-07	7.7575E-08	1.7735E-05
	0.75	1.5856E-03	4.7832E-04	9.1648E-05	1.1850E-05	7.2261E-07	1.4087E-04
	1.00	1.1054E-03	2.7337E-04	3.2725E-05	3.6311E-06	1.4436E-07	6.2178E-04
0.50	0.25	8.5638E-05	8.2546E-06	1.8044E-06	4.8721E-07	4.6598E-08	2.1230E-06
	0.50	7.9700E-04	1.6716E-04	3.0022E-06	1.4370E-06	3.1030E-07	7.0943E-05
	0.75	1.5856E-03	4.7832E-04	9.1648E-05	4.7401E-05	2.8905E-06	5.6348E-04
	1.00	1.1054E-03	2.7337E-04	3.2725E-05	1.4524E-05	5.7745E-07	2.4871E-03
0.75	0.25	8.5638E-05	8.2546E-06	1.8044E-06	1.0962E-06	1.0484E-07	4.7760E-06
	0.50	7.9700E-04	1.6716E-04	3.0022E-06	3.2333E-06	6.9817E-07	1.5962E-04
	0.75	1.5856E-03	4.7832E-04	9.1648E-05	1.0665E-04	6.5035E-06	1.2678E-03
	1.00	1.1054E-03	2.7337E-04	3.2725E-05	3.2680E-05	1.2993E-06	5.5960E-03

(a) 3D plots for different values of α (b) 3D plots for different values of α
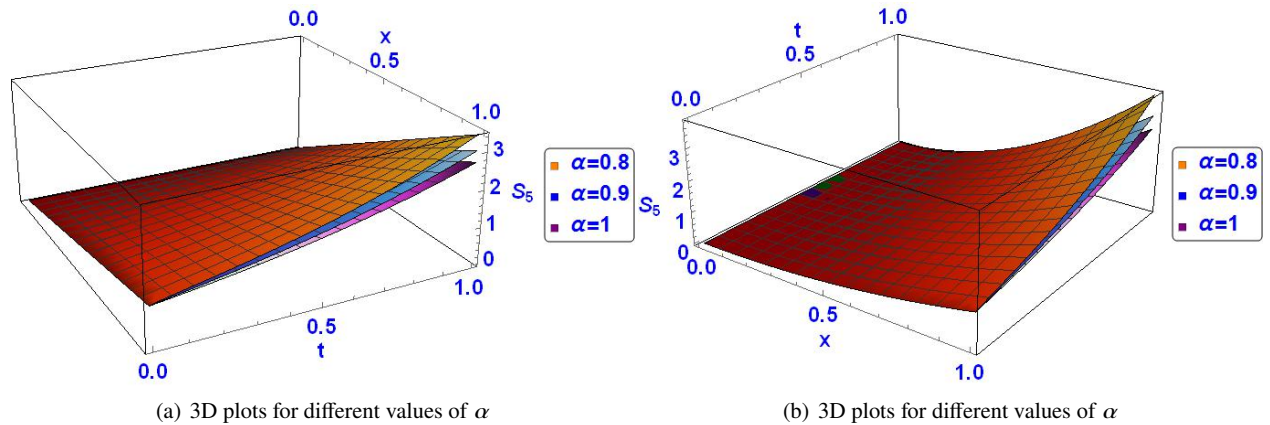
Fig. 3 Surface behavior in computational domain $x, t \in [0, 1]$ and $\alpha = 0.8, 0.9, 1$ in S_m ($m = 5$) solution of $_{TF}$GBE^{PD} (7) and $_{TF}$GBE^{PD} (11) respectively.

$$\phi_1(x,t) = -\frac{\hbar x^2 t^\alpha}{\Gamma(\alpha+1)}; \phi_2(x,t) = -\frac{\sqrt{\pi}8^{-\alpha}(2^\alpha - 4)\hbar^2 x^2 t^{2\alpha}}{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)} - \frac{\hbar x^2(\hbar+1)t^\alpha}{\Gamma(\alpha+1)};\ldots \tag{13}$$

In similar manner, one can get the values of $\phi_j(x,t)$ for $j \geq 3$. Fifth order approximate solution for the nonlinear $_{TF}$GBE^{PD} (11), in terms of controlling parameters $\hbar$ is as follows:

$$S_5(x,t,\hbar) = \phi_0(x,t) + \sum_{i=1}^{5} \phi_i(x,t). \tag{14}$$

Moreover, if $\hbar = -1, \alpha = 1$ the solution (14) reduces to $S_5(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!}\right)$. This obtained solution is similar to the exact solution $\phi(x,t) = x^2 e^t$, which is same as computed via DTM, HPM, FRDTM, AVIM and HPTM [12, 17, 18, 19, 21].

3 RESULTS AND DISCUSSIONS

For the nonlinear $_{TF}\text{GBE}^{PD}$ (7) and the nonlinear $_{TF}\text{GBE}^{PD}$ (11), the $\hbar$-curve for different values of $\alpha = 0.8, 0.9, 1$ for $x = t = 0.5$; 2D plots of the absolute errors in S_m $(m = 4, 5)$ solutions at $x = 0.5$ for $\alpha = 1$ are depicted in Figure 1 and Figure 2, respectively. The surface behavior of the S_m $(m = 5)$ solution for different values of $\alpha = 0.8, 0.9, 1.0$ are depicted in Figure 3. Moreover, the relative error $\left(\text{RE} = \left|\frac{S_5 - S_4}{S_5}\right|\right)$ in the S_m $(m = 4)$ solution for different values of $\alpha = 0.8, 0.9, 1.0$ and the absolute errors in S_m $(m = 4, 5)$ solutions corresponding to optimal value of $\hbar$ are reported in Table 1 and Table 2, respectively. From these tables, it is seen that the accuracy of S_m $(m = 4)$ solution is much better than the accuracy in HPM [17] results. Moreover, in the literature, it is seen that for given order of approximation, the HPM results are either equivalent or better than existing results computed via AVIM [18], HPTM [19] and Extended FRDTM [21] solutions. This concludes that the computed q-HAM results of examples (1),(2) at optimal values of $\hbar$ are more accurate as compared to the existing results in [18, 19, 21].

4 CONCLUSION

In the present work, q-HAM have been successfully implemented to study the nonlinear time fractional generalized Burger equation with proportional delay, considering fractional derivative of Caputo type. The computed results are analyzed in terms of absolute error (and relative error norms in the fourth order approximation) for two test problems of nonlinear $_{TF}\text{GBE}^{PD}$. From the findings of previous section, we reach at the conclusion that the evaluated q-HAM results are much better than the existing results via DTM, HPM, FRDTM, AVIM and HPTM [12, 17, 18, 19, 21], and these evaluated results approaches towards the exact solution very rapidly in comparison to the results evaluated via existing techniques.

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