Growth and Distortion Theorems for Some Univalent Harmonic Mappings

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ABSTRACT

Let S and K denote the usual classes of normalized univalent analytic and normalized convex analytic functions, respectively. Similarly, let S_H^0 and K_H^0, respectively, denote these classes in the harmonic case. It is known that the classes $S_H^o(S) = \{h+\overline{g} \in S_H^o : \quad h+e^{i\theta}g \in S \text{ for some } \theta \in \mathbb{R}\}$ and $K_H^0(K) = \{h+\overline{g} \in K_H^0 : \ h+e^{i\theta}g \in K \ \text{ for some } \ \theta \in \mathbb{R}\}$ are, respectively, subclasses of normalized univalent harmonic and normalized convex harmonic functions. We give estimates of some functionals defined on the functions of these classes.

1. Introduction and Preliminaries

A continuous complex-valued function $f = u + iv$ defined on a simply connected domain $D \subset \mathbb{C}$ is harmonic if u and v are real valued harmonic functions in D. Let $\mathcal{H}$ denote the class of all those harmonic functions f that are defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and are normalized by $f(0) = 0 = f_z(0) - 1$. Each $f \in \mathcal{H}$ admits the decomposition $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ with Taylor series expansion of the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

The functions h and g are called, respectively, analytic and co-analytic parts of f. Lewy's theorem [9] implies that every harmonic function f on $\mathbb{D}$ is locally univalent and sense-preserving if the Jacobian J_f of f, defined by: $J_f(z) = |h'(z)|^2 - |g'(z)|^2$, satisfies the condition $J_f(z) > 0$ on $\mathbb{D}$, or equivalently, $|w(z)| < 1$ in $\mathbb{D}$, where $w(z) = g'(z)/h'(z)$, $h'(z) \neq 0$ in $\mathbb{D}$, is called the second dilatation or complex dilatation of f. We denote by S_H the subclass of $\mathcal{H}$ consisting of all those functions that are univalent and sense-preserving on $\mathbb{D}$ and by S_H^0, the family of all $f \in S_H$ in which $f_{\overline{z}}(0) = 0$. Note that the familiar class S of all normalized analytic and univalent functions defined on $\mathbb{D}$ is contained in S_H^0. We denote by K_H^0, S_H^{*0} and C_H^0 the subclasses of S_H^0 whose functions map $\mathbb{D}$ onto, respectively, convex, starlike (with respect to the origin), and close-to-convex domains, just as K, S^* and C are the subclasses of S whose members map $\mathbb{D}$ onto these respective domains.

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In 1984, Clunie and Sheil-Small [3] constructed a harmonic function (popularly known as harmonic Koebe function) belonging to the class S_H^0, given by $K(z) = H(z) + \overline{G(z)}$, where

$$H(z) = \frac{z - \frac{z^2}{2} + \frac{z^3}{6}}{(1-z)^3} \quad \text{and} \quad G(z) = \frac{\frac{z^2}{2} + \frac{z^3}{6}}{(1-z)^3}. \tag{1.2}$$

and proposed the following conjecture.

Conjecture 1.1. *For all $f \in S_H^0$ having the series representation as in (1.1), we have*

$$|a_n| \leq A_n, \quad |b_n| \leq B_n, \quad \text{for all } n \geq 2$$

where, $A_n = \frac{(2n+1)(n+1)}{6}$ and $B_n = \frac{(2n-1)(n-1)}{6}$ are, respectively, the coefficients of $H(z)$ and $G(z)$ defined by (1.2).

Although, this conjecture has been settled for a number of subclasses of S_H^0 (See [3, 14, 15], 7, 11 and 12), but it is still open for the full class S_H^0. Recently, Ponnusamy and Kaliraj [10] introduced the following classes of univalent harmonic mappings.

$$S_H^o(S) = \{h + \overline{g} \in S_H^o : \ h + e^{i\theta} g \in S \text{ for some } \theta \in \mathbb{R}\}$$

and

$$S_H(S) = \{f = f_0 + b_1\overline{f_0} : \ f_0 \in S_H^0(S) \text{ and } b_1 \in \mathbb{D}\}.$$

Obviously, $S_H^0(S) \subseteq S_H^0$ and $S_H(S) \subset S_H$. The family $S_H^0(S)$ is a compact normal family. They proved that Conjecture 1.1 holds for functions in $S_H^0(S)$ and hence, in view of [14], for functions convex in one direction. As a consequence of this result they derived growth and covering theorems and sharp bounds on the Jacobian and curvature of f, $f \in S_H(S)$. In the same paper i.e [10], Ponnusamy and Kaliraj also proved analogous results for the classes

$$K_H^0(K) = \{h + \overline{g} \in K_H^0 : \ h + e^{i\theta} g \in K \ \text{ for some } \ \theta \in \mathbb{R}\},$$

and

$$K_H(K) = \{f = f_0 + b_1\overline{f_0} : \ f_0 \in K_H^0(K) \ \text{ and } \ b_1 \in \mathbb{D}\}.$$

In the present article, we establish estimates for some functionals involving functions from the class $S_H(S)$, $S_H^0(S)$, $K_H(K)$ and $K_H^0(K)$.

We shall need following definitions and results to prove our main results.

The classical Schwarz-Pick estimate for an analytic function w such that $|w(z)| < 1$ on $\mathbb{D}$ is the inequality

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2} \qquad (|z| < 1). \tag{1.3}$$

Ruscheweyh [13] obtained the best possible estimates on higher order derivatives of bounded analytic functions on the unit disk. Anderson and Rovnyak [1] also derived some similar estimates for different classes of analytic functions using other methods. In particular, they proved that if w is an analytic function and $|w(z)| < 1$ in $\mathbb{D}$, then

$$(1-|z|^2)^{n-1}\left|\frac{w^{(n)}(z)}{n!}\right| \leq \frac{1-|w(z)|^2}{1-|z|^2} \qquad (n=1,2,......) \tag{1.4}$$

The case $z = 0$ (1.5) leads to the classical result (See [2, 8]): If

$$w(z) = c_0 + c_1 z + c_2 z^2 + \cdots \tag{1.5}$$

is analytic and $|w(z)| < 1$ in $\mathbb{D}$, then

$$|c_n| \leq 1 - |c_0|^2, \tag{1.6}$$

for every $n \geq 1$.

2. Main Results

Ponnusamy and Kaliraj [10] established following inequalities which we shall need in the present section.

Lemma 2.1. [10] *Every function $f \in S_H^0(S)$ satisfies the inequalities*

$$\frac{1}{6}\left[1-\left(\frac{1-r}{1+r}\right)^3\right] \leq |f(z)| \leq \frac{1}{6}\left[\left(\frac{1+r}{1-r}\right)^3 - 1\right], \qquad r = |z| < 1.$$

The above inequalities are sharp and the equality is attained for the harmonic Koebe function K defined by (1.2) and its rotations.

Lemma 2.2. [10] *Let $f \in S_H(S)$ with $b_1 = f_{\overline{z}}(0)$. Then for any z with $0 < |z| = r < 1$ the inequalities*

$$|h'(z)| \leq (1 + r|b_1|)\frac{(1+r)}{(1-r)^4} \qquad \text{and} \qquad |g'(z)| \leq (r + |b_1|)\frac{(1+r)}{(1-r)^4}$$

hold. These bounds are sharp. The equality is attained for the close-to-convex functions $f(z) = K(z) + \overline{b_1 K(z)}$, where K is the harmonic Koebe function defined by (1.2).

We now prove the following results.

Theorem 2.3. *If $f \in S_H^0(S)$, $f = h + \overline{g}$ and $|z| = r < 1$, then*
(i) $|h| - |g| \leq \min\left\{\frac{r}{(1-r)^2}, \frac{1}{6}\left[\left(\frac{1+r}{1-r}\right)^3 - 1\right]\right\}$ *(ii)* $|h| + |g| \geq \max\left\{\frac{r}{(1+r)^2}, \frac{1}{6}\left[1 - \left(\frac{1-r}{1+r}\right)^3\right]\right\}$.

Proof. For $f = h + \overline{g}$, it follows from $f \in S_H^0(S)$ that there must exist at least one $\theta \in \mathbb{R}$ such that $h + e^{i\theta}g \in S$. Applying growth theorem (see [5]) for the class S on $h + e^{i\theta}g$, we get

$$\frac{r}{(1+r)^2} \leq |h + e^{i\theta}g| \leq \frac{r}{(1-r)^2} \tag{2.1}$$

Also,

$$|h| - |g| \leq |h + e^{i\theta}g| \leq |h| + |g|. \tag{2.2}$$

From Lemma 2.1 and (2.2), we have following

$$|h| - |g| \leq \frac{1}{6}\left[\left(\frac{1+r}{1-r}\right)^3 - 1\right] \tag{2.3}$$

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and

$$|h| + |g| \geq \frac{1}{6}\left[1 - \left(\frac{1-r}{1+r}\right)^3\right]. \tag{2.4}$$

It follows from (2.1) and (2.2) that

$$|h| - |g| \leq \frac{r}{(1-r)^2} \tag{2.5}$$

$$|h| + |g| \geq \frac{r}{(1+r)^2}. \tag{2.6}$$

Now, (2.3) and (2.5) imply

$$|h| - |g| \leq \min\left\{\frac{r}{(1-r)^2}, \frac{1}{6}\left[\left(\frac{1+r}{1-r}\right)^3 - 1\right]\right\}.$$

Similarly, (2.4) and (2.6) give

$$|h| + |g| \geq \max\left\{\frac{r}{(1+r)^2}, \frac{1}{6}\left[1 - \left(\frac{1-r}{1+r}\right)^3\right]\right\}$$

as desired. □

Theorem 2.4. *For each $f \in S_H^0(S)$, $f = h + \overline{g}$ and $|z| = r < 1$, we have*

$$|h'| - |g'| \leq \frac{1+r}{(1-r)^3} \tag{2.7}$$

$$|h'| + |g'| \geq \frac{1-r}{(1+r)^3} \tag{2.8}$$

*

The proof easily follows from Theorem 2.3 by applying the distortion theorem (see [5]) for the class S instead of growth theorem.

Following results for the class $K_H^0(K)$ of harmonic convex functions follow similarly.

Theorem 2.5. *For $f \in K_H^0(K)$, $f = h + \overline{g}$ and $|z| = r < 1$, we have (i) $|h| - |g| \leq \frac{r}{1-r}$ (ii) $|h| + |g| \geq \frac{r}{1+r}$.*

Theorem 2.6. *If $f \in K_H^0(K)$, $f = h + \overline{g}$ and $|z| = r < 1$, then (i) $|h'| - |g'| \leq \frac{1}{(1-r)^2}$ (ii) $|h'| + |g'| \geq \frac{1}{(1+r)^2}$.*

Theorem 2.7. *For each $f \in S_H^0(S)$, $f = h + \overline{g}$ and $|z| = r < 1$, one has*

$$|g(z)| \geq \left|\frac{1 + 3r + 5(1-r)^3}{(1-r)^3} - \frac{r^2 + r + 1}{(1+r)^2}\right| \tag{2.9}$$

and

$$|g(z)| \leq \frac{(1-r)\left(5(1-r)^2 + 6r(1-r) - 9\right) + 4}{6(1-r)^3}. \tag{2.10}$$

Proof. Taking $|b_1| = 0$ in Lemma 2.2, we get

$$|h'(z)| \leq \frac{1+r}{(1-r)^4} \tag{2.11}$$

and

$$|g'(z)| \leq \frac{r(1+r)}{(1-r)^4} \tag{2.12}$$

The right hand side of (2.10), is obtained immediately by integrating (2.12) along a radial line $\zeta = te^{i\theta}$. In order to prove right hand side of (2.9), we first note that g is univalent. Let $\tau = g(\{z : |z| = r\})$ and let $\xi \in \tau$ be the nearest point to the origin. By a rotation we may assume that $\xi > 0$. Let γ be the line segment $0 \leq \xi \leq \xi_1$ and suppose that $z_1 = g^{-1}(\xi_1)$ and $L = g^{-1}(\gamma)$. With ζ as the variable of integration on L we have $d\xi = g'(\zeta)d\zeta > 0$ on L. Hence, using (2.8)

$$\begin{aligned}\xi_1 = \int_0^{\xi} d\xi = \int_0^{z_1} g'(\zeta)d\zeta &\geq \int_0^{z_1} |g'(\zeta)||d\zeta| \geq \int_0^{r} |g'(te^{i\theta})|dt \\ &\geq \int_0^{r} \left| \frac{1-r}{(1+r)^3} - \frac{(1+r)}{(1-r)^4} \right| dr \\ &= \left| \frac{1+3r+5(1-r)^3}{(1-r)^3} - \frac{r^2+r+1}{(1+r)^2} \right|. \end{aligned}$$

□

The following result of Ponnusamy and Kaliraj [10] is required in our next result.

Lemma 2.8. [10] *Suppose that $f = h + \overline{g} \in S_H(S)$ with the series representation as in (1.1). Then*

$$|a_n| < \frac{1}{3}(2n^2+1) \quad \text{and} \quad |b_n| < \frac{1}{3}(2n^2+1) \quad for\ \ all\ \ n \geq 2.$$

Theorem 2.9. *For $f \in S_H$, $f = h + \overline{g}$ with $|b_1| = \alpha \in (0,1)$ and $|z| = r < 1$, we have*

$$(i) \left| \frac{zg''(z)}{g'(z)} \right| \geq \frac{2r^2-6r}{1-r^2} - \frac{r(1-\alpha^2)}{|r-\alpha|(1-r\alpha)},$$

$$(ii) \frac{r(\alpha^2-1)}{|\alpha-r|(1-\alpha r)} + \frac{r^2-6r+1}{1-r^2} \leq \Re\left(1 + \frac{zg''(z)}{g'(z)}\right) \leq \frac{r(1-\alpha^2)}{|r-\alpha|(1-\alpha r)} + \frac{r^2+6r+1}{1-r^2}.$$

Proof. Let $f = h + \overline{g}$ and fix $t \in \mathbb{D}$. We apply the disk automorphism and obtain

$$F(z) = \frac{f\left(\frac{z+t}{1+\bar{t}z}\right) - f(t)}{(1-|t|^2)h'(t)} = H(z) + \overline{G(z)},$$

which is again in S_H. We take

$$H(z) = z + A_2(t)z^2 + A_3(t)z^3 + \cdots,$$

After a simple calculation we get

$$A_2(t) = \frac{1}{2}\{(1-|t|^2)\frac{h''(t)}{h'(t)} - 2\bar{t}\},$$

Taking $n = 2$ in Lemma 2.8, we get $|A_2(t)| < 3$. Therefore

$$\frac{2r^2-6r}{1-r^2} \leq \Re\left(\frac{zh''(z)}{h'(z)}\right) \leq \frac{2r^2+6r}{1-r^2}, \qquad |z| = r < 1. \tag{2.13}$$

Using the relation $g' = wh'$ we obtain

$$\frac{zg''(z)}{g'(z)} = \frac{zw'(z)}{w(z)} + \frac{zh''(z)}{h'(z)}. \tag{2.14}$$

Thus, w satisfies (See [6, p.118])

$$\left|\frac{w(z)-w(0)}{1-\overline{w(0)}w(z)}\right| \leq |z| \qquad (|z|=r), \tag{2.15}$$

from which it follows that

$$\left|w(z)-\frac{w(0)(1-r^2)}{1-|w(0)|^2r^2}\right| \leq \frac{r(1-|w(0)|^2)}{1-|w(0)|^2r^2}. \tag{2.16}$$

Here note that $|w(0)|=|c_0|=|b_1|=\alpha$, so that, by (2.16), we have

$$\frac{|r-\alpha|}{1-\alpha r} \leq |w(z)| \leq \frac{r+\alpha}{1+\alpha r}. \tag{2.17}$$

Taking into account (2.13), (2.14), (2.17) and the Schwarz-Pick inequality (1.4), we obtain for $|z|=r<1$,

$$\frac{zg''(z)}{g'(z)} = \frac{zw'(z)}{w(z)} + \frac{zh''(z)}{h'(z)}$$

and so,

$$\begin{aligned}\left|\frac{zg''(z)}{g'(z)}\right| &\geq \left|\frac{zh''(z)}{h'(z)}\right| - \left|\frac{zw'(z)}{w(z)}\right| \\ &\geq \frac{2r^2-6r}{1-r^2} - \frac{r(1-|w(z)|^2)}{|w(z)|(1-r^2)} \\ &\geq \frac{2r^2-6r}{1-r^2} - \frac{r(1-r^2)(1-\alpha^2)}{(1-r^2)|r-\alpha|(1-\alpha r)} \\ &= \frac{2r^2-6r}{1-r^2} - \frac{r(1-\alpha^2)}{|r-\alpha|(1-\alpha r)}.\end{aligned}$$

Moreover,

$$1+\frac{zg''(z)}{g'(z)} = \frac{zw'(z)}{w(z)} + 1 + \frac{zh''(z)}{h'(z)}, \quad gives$$

$$\Re\left(1+\frac{zg''(z)}{g'(z)}\right) = \Re\frac{zw'(z)}{w(z)} + \Re\left(1+\frac{zh''(z)}{h'(z)}\right).$$

By using, (1.4), (2.13) and (2.17), we have

$$\Re\left(1+\frac{zg''(z)}{g'(z)}\right) \geq \frac{r(\alpha^2-1)}{|\alpha-r|(1-\alpha r)} + \frac{r^2-6r+1}{1-r^2}.$$

Similarly, we have

$$\Re\left(1+\frac{zg''(z)}{g'(z)}\right) \leq \frac{r(1-\alpha^2)}{|r-\alpha|(1-\alpha r)} + \frac{r^2+6r+1}{1-r^2},$$

as asserted. □

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