On Symmetry Analysis in Finding Solutions of the One Dimensional Wave Equation

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ABSTRACT

In this paper we obtain the Lie invariance condition for second order partial differential equations. This condition is used to obtain the determining equations of the 1-dimensional wave equation with constant speed. The determining equations are split to obtain an overdetermined system of partial differential equations which are solved to obtain the symmetries of the wave equation. By making an appropriate transformation between the dependent and independent variable, the wave equation is reduced to an easily solvable ordinary differential equation. We solve this resulting differential equation to obtain the solutions of the wave equation. In particular, the one dimensional wave equation with unit speed has been solved.

1 Introduction

Over a hundred years ago, Sophus Lie introduced the theory of Lie symmetry groups which are invertible point transformations of the independent and dependent variables in differential equations. In such group analysis, these transformations lead the original differential equation of first order (which may be a complicated one) to becoming a separable one, which can be easily solved. In addition, for higher order differential equations, group analysis of the equation is used to reduce the order of the differential equation by one. A lot of literature on group analysis for ordinary and partial differential equations can be found in [2, 5].

We call transformations leaving objects invariant or unchanged as *Symmetries.* In [16] it is pointed out that symmetries account for various laws in nature, A very important implication of symmetry in physics and mathematics is the existence of conservation laws. This is because their reproduction at different times and places depend on invariance laws. In this connection Emma N*ö*ether [14] observed and proved a relation between continuous symmetries and conservation laws. Scientists like Kepler got interested in studying motion of planets using symmetries while Newton studied the orbits of planets using the laws of mechanics as a symmetry principle.

Partial differential equations are of paramount importance in particle Physics, nonlinear optics, fluid dynamics in addition to its applications in general relativity to differential and algebraic geometry, topology, etc. Research dedicated to using symmetry analysis for partial differential equations arising in Mathematical Physics may be found in [20, 13]. In addition symmetry analysis for time fractional partial differential equations can be found in [8]. Lie symmetry analysis for the Kudryashov-Sinelshikov equation may be found in [23] while Lie symmetry analysis for the Burgers' equation may be found in [11, 12]. Studies on the Korteweg-de Vries equation using Lie symmetry analysis are seen in [10, 22]. Interests in studying symmetries is so large that in [19] the definition of an admitted Lie group is developed for Stochastic differential equations.

We now state the definition of a Lie group which we will be using. We will also give some examples of Lie group to illustrate the definition.

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Definition 1 ([3]) In general, we consider transformations $\bar{t}_i = f_i(t_j, \delta), i, j = 1, 2, \cdots, n$ which continuously depend on the parameter δ. Further we assume that, for each i, f_i is a smooth function of the variables t_j and has convergent Taylor series in δ.
These set of transformations are said to form a group if:

1. Two transformations carried out in succession are equivalent to another transformation of the set
2. There is a transformation for which the source and image points coincide.
3. Each transformation has an inverse.

Remark 1 The associativity law for groups follows from the *closure* property.

Remark 2 In general, the order in which the transformations are carried out matters. If the order does not matter, then we label the group as **abelian**.

Example 1 $\bar{t} = a^{\delta} t, t \in \mathbb{R} \backslash \{0\}$ is a one parameter group called the *stretching group*.

Example 2 $\bar{t} = t + \delta$, which is also a group called the *translational group*.

Example 3 $\bar{t}_1 = t_1 \cos\delta - t_2 \sin\delta, \bar{t}_2 = t_1 \sin\delta + t_2 \cos\delta$ is known as a *Rotational group*.

In this paper we perform group analysis of the one-dimensional wave equation, which is of the form,

$$\frac{\partial^2 u}{\partial t^2}(t, x) - c^2 \frac{\partial^2 u}{\partial x^2}(t, x) = 0, \tag{1}$$

where u is a real valued function defined on $I \times D$, and where I is an open interval in $\mathbb{R}$ and D is an open set in $\mathbb{R}$. Equation (1) under study is a second order partial differential equation with c as the constant speed.
Wave equations find applications in modeling the air column of a clarinet or organ pipe, modeling tension via springs, motion of a vibrating string, study of damping, elastic waves in a rod, acoustic model for seismic waves, sound waves in liquids and gases, etc.
Traditional methods to solve wave equations are inverse scattering method [1], Hirota bilinear method [9], Darboux transformation [6], B$\ddot{a}$ckund method [4] and Painleve's extension method [7]. Other methods of solution using D'Alembert's formula and method of separation of variables can be found in [17, 18]. More studies for the wave equation can be found in [21].
We have used Taylor's theorem for a function of several variables to obtain a Lie type invariance condition for second order partial differential equations. Using this we obtain symmetries and hence the solutions of the wave equation, for which we have not found any literature.

2 Lie Invariance Condition Second Order Partial Differential Equations

Let $u = u(t, x)$. Then we consider transformations with g_1, g_2, g_3 smooth functions in t, x, u having convergent Taylor series in δ which are of the form,

$$\begin{cases} \bar{t} = g_1(t, x, u) = t + \delta\tau(t, x, u) + O(\delta^2), \\ \bar{x} = g_2(t, x, u) = x + \delta\xi(t, x, u) + O(\delta^2), \\ \bar{u} = g_3(t, x, u) = u + \delta\eta(t, x, u) + O(\delta^2). \end{cases} \tag{2}$$

where $\tau(t, x, u) = \left.\frac{\partial g_1}{\partial \delta}\right|_{\delta=0}, \quad \xi(t, x, u) = \left.\frac{\partial g_2}{\partial \delta}\right|_{\delta=0}, \quad \eta(t, x, u) = \left.\frac{\partial g_3}{\partial \delta}\right|_{\delta=0}.$
In order to calculate the prolongation of a given transformation, we need to differentiate (2) with respect to each of the

parameters t and x. To do this we introduce the following *total derivatives*:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots , \tag{3}$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots . \tag{4}$$

The first two equations of (2) may be inverted (locally) to give t and x in terms of $\bar{t}$ and $\bar{x}$, provided that the Jacobian is non-zero, that is,

$$\mathcal{J} = \begin{vmatrix} D_t\bar{t} & D_t\bar{x} \\ D_x\bar{t} & D_x\bar{x} \end{vmatrix} \neq 0, \quad \text{where} \quad u = u(x,t). \tag{5}$$

Since the last equation of (2) contains $\bar{u}$ as a function of some variables, one of which is $u(t,x)$, we see that if equation (5) is satisfied, then by local inverse function theorem, the last equation of (2) can be rewritten as

$$\bar{u} = \bar{u}(\bar{t}, \bar{x}). \tag{6}$$

Applying the chain rule to equation (6), we obtain,

$$\begin{bmatrix} D_t\bar{u} \\ D_x\bar{u} \end{bmatrix} = \begin{bmatrix} D_t\bar{t} & D_t\bar{x} \\ D_x\bar{t} & D_x\bar{x} \end{bmatrix} \begin{bmatrix} \bar{u}_{\bar{t}} \\ \bar{u}_{\bar{x}} \end{bmatrix},$$

and therefore by (Cramer's rule)

$$\bar{u}_{\bar{t}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u} & D_t\bar{x} \\ D_x\bar{u} & D_x\bar{x} \end{vmatrix}, \quad \bar{u}_{\bar{x}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{t} & D_t\bar{u} \\ D_x\bar{t} & D_x\bar{u} \end{vmatrix} \tag{7}$$

Equation (7) can be simplified to get the extended infinitesimal representation,

$$\bar{u}_{\bar{t}} = u_t + \delta\eta_{[t]} + O(\delta^2), \quad \bar{u}_{\bar{x}} = u_x + \delta\eta_{[x]} + O(\delta^2), \tag{8}$$

where

$$\eta_{[t]} = D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau), \tag{9}$$

$$\eta_{[x]} = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau). \tag{10}$$

The explicit expression for equation (10) is

$$\eta_{[t]} = \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2,$$

$$\eta_{[x]} = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t.$$

Continuing the procedure, we can obtain the second-order prolongations as follows:

$$\bar{u}_{\overline{tt}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\bar{t}} \\ D_x\bar{x} & D_x\bar{u}_{\bar{t}} \end{vmatrix}, \quad \bar{u}_{\overline{xx}} = \frac{1}{\mathcal{J}} \begin{vmatrix} D_t\bar{u}_{\bar{x}} & D_t\bar{t} \\ D_x\bar{u}_{\bar{x}} & D_x\bar{t} \end{vmatrix} \tag{11}$$

$$\bar{u}_{\overline{xt}} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{x} & D_t\bar{u}_{\bar{x}} \\ D_x\bar{x} & D_x\bar{u}_{\bar{x}}\end{vmatrix} = \frac{1}{\mathcal{J}}\begin{vmatrix} D_t\bar{u}_{\bar{t}} & D_t\bar{t} \\ D_x\bar{u}_{\bar{t}} & D_x\bar{t}\end{vmatrix} \tag{12}$$

On simplifying (11) and (12) we get the extended infinitesimal representations, namely

$$\bar{u}_{\overline{tt}} = u_{tt} + \delta\eta_{[tt]} + O(\delta^2), \quad \bar{u}_{\overline{xx}} = u_{xx} + \delta\eta_{[xx]} + O(\delta^2), \quad \bar{u}_{\overline{tx}} = u_{tx} + \delta\eta_{[tx]} + O(\delta^2), \tag{13}$$

where

$$\eta_{[tt]} = D_t(\eta_{[t]}) - u_{tx}D_t(\xi) - u_{tt}D_t(\tau), \quad \eta_{[xx]} = D_x(\eta_{[x]}) - u_{xx}D_x(\xi) - u_{tx}D_x(\tau), \tag{14}$$

and,

$$\begin{aligned}\eta_{[tx]} &= D_t(\eta_{[x]}) - u_{xx}D_t(\xi) - u_{tx}D_t(\tau) \\ &= D_x(\eta_{[t]}) - u_{tx}D_x(\xi) - u_{tt}D_x(\tau)\end{aligned} \tag{15}$$

The explicit expressions for $\eta_{[tt]}, \eta_{[xx]}, \eta_{[tx]}$ given by equations (14) and (15) are

$$\begin{aligned}\eta_{[tt]} = \eta_{tt} - \xi_{tt}u_x + (2\eta_{tu} - \tau_{tt})u_t - 2\xi_{tu}u_xu_t + (\eta_{uu} - 2\tau_{tu})u_t^2 - \xi_{uu}u_xu_t^2 \\ - \tau_{uu}u_t^3 - 2\xi_t u_{xt} - 2\xi_u u_t u_{xt} + (\eta_u - 2\tau_t)u_{tt} - \xi_u u_x u_{tt} - 3\tau_u u_t u_{tt},\end{aligned} \tag{16}$$

$$\begin{aligned}\eta_{[xx]} = \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_xu_t - \xi_{uu}u_x^3 \\ - \tau_{uu}u_x^2u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt},\end{aligned} \tag{17}$$

$$\begin{aligned}\eta_{[xt]} = \eta_{xt} + (\eta_{tu} - \xi_{xt})u_x + (\eta_{xu} - \tau_{xt})u_t - \xi_{tu}u_x^2 + (\eta_{uu} - \xi_{xu} - \tau_{tu})u_xu_t \\ - \tau_{xu}u_t^2 - \xi_{uu}u_x^2u_t - \tau_{uu}u_xu_t^2 - \xi_t u_{xx} - \xi_u u_t u_{xx} + (\eta_u - \xi_x - \tau_t)u_{xt} \\ - 2\xi_u u_x u_{xt} - 2\tau_u u_t u_{xt} - \tau_x u_{tt} - \tau_u u_x u_{tt}.\end{aligned}$$

If $G : I \times D^7 \to \mathbb{R}$, is a differentiable function where I is an interval in $\mathbb{R}$ and D is an open set in $\mathbb{R}$, then for invariance, we need to have,

$$\begin{aligned} 0 &= G(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}}, \bar{u}_{\overline{tt}}, \bar{u}_{\overline{xx}}, \bar{u}_{\overline{tx}}) \\ &= G(t + \delta\tau + O(\delta^2), x + \delta\xi + O(\delta^2), u + \delta\eta + O(\delta^2), u_t + \delta\eta_{[t]} + O(\delta^2), \\ &\quad u_x + \delta\eta_{[x]} + O(\delta^2), u_{tt} + \delta\eta_{[tt]} + O(\delta^2), u_{xx} + \delta\eta_{[xx]} + O(\delta^2), \\ &\quad u_{tx} + \delta\eta_{[tx]} + O(\delta^2)) \\ &= G(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) + \delta\Big(\tau G_t + \xi G_x + \eta G_u + \eta_{[t]}G_{u_t} + \eta_{[x]}G_{u_x} \\ &+ \eta_{[tt]}G_{u_{tt}} + \eta_{[xx]}G_{u_{xx}} + \eta_{[tx]}G_{u_{tx}}\Big) + O(\delta^2)\end{aligned}$$

Equating the coefficient of δ, we get

$$\tau G_t + \xi G_x + \eta G_u + \eta_{[t]}G_{u_t} + \eta_{[x]}G_{u_x} + \eta_{[tt]}G_{u_{tt}} + \eta_{[xx]}G_{u_{xx}} + \eta_{[tx]}G_{u_{tx}} = 0$$

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The infinitesimal generator of the admitted group for the equation given by (17) is,

$$\zeta^* = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}.$$

The extension is given by,

$$\zeta^{(1)} = \tau G_t + \xi G_x + \eta G_u + \eta_{[t]}G_{u_t} + \eta_{[x]}G_{u_x} + \eta_{[tt]}G_{u_{tt}} + \eta_{[xx]}G_{u_{xx}} + \eta_{[tx]}G_{u_{tx}}. \quad (18)$$

The Lie type invariance condition is given by $\zeta^{(1)}G\mid_{G=0}= 0$, where $G(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx}) = 0$.

3 Symmetry Analysis of the Wave Equation

We shall obtain symmetries and hence the solutions of

$$u_{tt}(t,x) - c^2u_{xx}(t,x) = 0 \quad (19)$$

The Lie type invariance condition for the wave equation (19) gives

$$\eta_{[tt]} - c^2\eta_{[xx]} = 0. \quad (20)$$

Substituting the values of $\eta_{[tt]}$ and $\eta_{[xx]}$ from equations (16) and (17), we get the determining equations given by

$$\begin{aligned}\eta_{tt} - c^2\eta_{xx} - (\xi_{tt} + c^2(2\eta_{xu} - \xi_{xx}))u_x + (2\eta_{tu} - \tau_{tt} + c^2\tau_{xx})u_t - c^2(\eta_{uu} - 2\xi_{xu})u_x^2\\ - 2(\xi_{tu} + \tau_{xu})u_xu_t + (\eta_{uu} - 2\tau_{ut})u_t^2 + c^2\xi_{uu}u_x^3 + c^2\tau_{uu}u_x^2u_t - \xi_{uu}u_xu_t^2 - \tau_{uu}u_t^3\\ - c^2(\eta_u - 2\xi_x)u_{xx} + (2c^2\tau_x - 2\xi_t)u_{xt} + c^2(\eta_u - 2\tau_t)u_{xx} + 3c^2\xi_uu_xu_{xx} + c^2\tau_uu_tu_{xx}\\ + 2c^2\tau_uu_xu_{tx} - 2\xi_uu_tu_{xt} - c^2\xi_uu_xu_{xx} - 3c^2\tau_uu_tu_{xx} = 0.\end{aligned} \quad (21)$$

Splitting equation (21) with respect to u_x, we get

$$-(\xi_{tt} + c^2(2\eta_{xu} - \xi_{xx})) = 0, \quad (22)$$

Splitting equation (21) with respect to u_t, we get

$$2\eta_{tu} - \tau_{tt} + c^2\tau_{xx} = 0, \quad (23)$$

Splitting equation (21) with respect to u_x^2, we get

$$-c^2(\eta_{uu} - 2\xi_{xu}) = 0, \quad (24)$$

Splitting equation (21) with respect to u_xu_t, we get

$$-2(\xi_{tu} + \tau_{xu}) = 0, \quad (25)$$

Splitting equation (21) with respect to u_t^2, we get

$$\eta_{uu} - 2\tau_{ut} = 0, \quad (26)$$

Splitting equation (21) with respect to u_x^3, we get

$$c^2\xi_{uu} = 0, \tag{27}$$

Splitting equation (21) with respect to $u_x^2 u_t$, we get

$$c^2\tau_{uu} = 0, \tag{28}$$

Splitting equation (21) with respect to $u_x u_t^2$, we get

$$-\xi_{uu} = 0, \tag{29}$$

Splitting equation (21) with respect to u_t^3, we get

$$-\tau_{uu} = 0, \tag{30}$$

Splitting equation (21) with respect to u_{xx}, we get

$$c^2(\xi_x - \tau_t) = 0, \tag{31}$$

Splitting equation (21) with respect to u_{xt}, we get

$$c^2\tau_x - \xi_t = 0, \tag{32}$$

Splitting equation (21) with respect to $u_x u_{xx}$, we get

$$2c^2\xi_u = 0, \tag{33}$$

Splitting equation (21) with respect to $u_t u_{xx}$, we get

$$-2c^2\tau_u = 0 = 0, \tag{34}$$

Splitting equation (21) with respect to $u_x u_{tx}$, we get

$$2c^2\tau_u = 0 = 0, \tag{35}$$

Splitting equation (21) with respect to $u_t u_{xt}$, we get

$$-2\xi_u = 0 = 0, \tag{36}$$

Splitting equation (21) with respect to the constant term, we get

$$\eta_{tt} - c^2\eta_{xx} = 0. \tag{37}$$

From equations (33), (34) and hence (26), we get,

$$\tau(t,x,u) = A(t,x), \quad \xi(t,x,u) = B(t,x), \quad \eta(t,x,u) = P(t,x)u + Q(t,x),$$

where A, B, P are arbitrary functions of t and x.
From equations (31), (32), (22) and (23), we get, $P_t = 0, \quad P_x = 0 \Rightarrow P(t,x) = p$, a constant.
Hence, $U = pu + Q(t,x)$, where Q satisfies the wave equation.
The wave equation admits a four dimensional Lie group with generators,

$$\zeta_1^* = A(t,x)\frac{\partial}{\partial t}, \quad \zeta_2^* = B(t,x)\frac{\partial}{\partial x}, \quad \zeta_3^* = u\frac{\partial}{\partial u}, \quad \zeta_4^* = Q(t,x)\frac{\partial}{\partial u}.$$

We now seek a solution of the wave equation by making a special choice of the infinitesimals namely, $A = x, \quad B = t, \quad Q = 0$.
The associated invariant surface condition is
$xu_x + tu_t = pu$, which is solved to get $u = x^p F\left(\frac{t}{x}\right)$, where F is an arbitrary function.
Substituting in the 1-dimension wave equation, we get,

$$(c^2r^2 - 1)F''(r) - 2c^2r(p-1)F'(r) + c^2p(p-1)F(r) = 0 \quad \text{where} \quad r = \frac{t}{x}. \tag{38}$$

This can be integrated easily giving,

$$F(r) = \begin{cases} \dfrac{c_1}{c_2}log\left(\dfrac{rc-1}{rc+1}\right) & \text{if } p = 0 \\ c_1\left(r - \dfrac{1}{c}\right)^p + c_2\left(r + \dfrac{1}{c}\right)^p & \text{if } p \neq 0 \end{cases}$$

If $p = 0$, exact solution is

$$u(x,t) = \frac{c_1}{c^2}log\left(\frac{ct-x}{ct+x}\right) + c_2 \tag{39}$$

If $p \neq 0$, exact solution is

$$u(x,t) = x^p\left[c_1F_1\left(\frac{t}{x}\right) + c_2F_2\left(\frac{t}{x}\right)\right] \tag{40}$$

where $Fi, i = 1, 2$ are solutions of equation (38).

The solutions of equation (38) for some integer values of p are presented in table 1 below:

Table 1 Solutions of equation (38) for some integer values of p.

p	F_1	F_2
1	1	r
2	$\frac{c^2r^2-2cr+1}{c^2}$	$\frac{c^2r^2+2cr+1}{c^2}$
3	$\frac{c^3r^3-3c^2r^2+3cr-1}{c^3}$	$\frac{c^3r^3+3c^2r^2+3cr+1}{c^3}$
4	$\frac{c^4r^4-4c^3r^3+6c^2r^2-4cr+1}{c^4}$	$\frac{c^4r^4+4c^3r^3+6c^2r^2+4cr+1}{c^4}$
-1	$\frac{c}{cr-1}$	$\frac{c}{cr+1}$
-2	$\frac{c^2}{c^2r^2-2cr+1}$	$\frac{c^2}{c^2r^2+2cr+1}$
-3	$\frac{c^3}{c^3r^3-3c^2r^2+3cr-1}$	$\frac{c^3}{c^3r^3+3c^2r^2+3cr+1}$
-4	$\frac{c^4}{c^4r^4-4c^3r^3+6c^2r^2-4cr+1}$	$\frac{c^4}{c^4r^4+4c^3r^3+6c^2r^2+4cr+1}$

We shall conclude the analysis for the wave equation by giving as an example, the particular solution of hyperbolic wave equation with unit speed.

Example 4 Consider the hyperbolic wave equation with unit speed given by $u_{tt}(t,x) - u_{xx}(t,x) = 0$.
To obtain a particular solution, consider, $A = t, \quad B = x, \quad P = Q = 0$.
The associated invariant surface condition is $tu_x + xu_t = 0$
The solution of this linear partial differential equation is $u(x,t) = F(t^2 - x^2)$, where F is arbitrary.
To find F, we use the wave equation $u_{tt} = u_{xx}$, and get $rF''(r) + F'(r) = 0$, where $r = t^2 - x^2$.
The solution of this Clairaut differential equation is
$F(r) = c_3 ln(r) + c_4$, where c_3, c_4 are arbitrary constants.
Re substituting r, we get,

$$u(x,t) = c_3 ln(t^2 - x^2) + c_4, \tag{41}$$

which is the exact solution of the 1-dimensional unit speed wave equation.

4 Conclusion

In this paper, we have performed symmetry analysis of the 1-dimensional wave equation with constant speed and have obtained its equivalents symmetries and explicit solutions.

1. The wave equation (19) admits the general infinitesimal generator of the Lie group given by

$$\zeta^* = A(t,x)\frac{\partial}{\partial t} + B(t,x)\frac{\partial}{\partial x} + (pu + Q(t,x))\frac{\partial}{\partial u}.$$

2. The wave equation with general constant speed admits solutions (found by symmetry analysis) given by equations (39) and (40).
3. The wave equation with unit speed admits solutions (found by symmetry analysis) given by equation (41).

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