Range and Null Space of Weighted Composition Operators on l^p Spaces

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ABSTRACT

Let l^p ($1 \leq p \leq \infty$) be the Banach space of all p-summable sequences (bounded sequences for p = 1) of complex numbers under the standard p-norm on it and C_ϕ be a composition operator on l^p induced by a function ϕ on **N** into itself. In this paper we discuss range and null space of weighted composition operators on l^p spaces.

Keywords: Range, Null, Weighted Composition operator.

RESULTS

Preposition 1: If u is a bounded away from zero then R (uc_ϕ) is closed.

Proof: Suppose u is a bounded away from zero. Let a > 0 such that $0 < \dfrac{1}{|u(n)|} \leq a$ for each $n \geq 1$.

$$\begin{aligned} f \in R(uc_\phi) &\Leftrightarrow f = (uc_\phi)(g) \text{ for some } g \text{ belongs to } l^p. \\ &\Leftrightarrow f(n) = u(n)\, g(\phi(n)) \text{ for each } n \geq 1. \\ &\Leftrightarrow \frac{f(n)}{u(n)} = g(\phi(n)) \text{ for each } n \geq 1. \\ &\Leftrightarrow \frac{f}{u} \in R(c_\phi) \\ &\Leftrightarrow \left(\frac{f}{u}\right) / A_n \text{ is constant [24].} \end{aligned}$$

Let $\left(f_m\right)_{m=1}^{\infty}$ be a sequence in R (uc_ϕ) for each $f_m \to f$ as $m \to \infty$.

Then by definition of convergence $\|f_m - f\|_p \to 0$ as $m \to \infty$. This implies that for each $n \geq 1$, $|f_m(n) - f(n)| \to 0$ as $m \to \infty$.

Now, $$\left|\frac{1}{u(n)} f_m(n) - \frac{1}{u(n)} f(n)\right| = \left|\frac{1}{u(n)}\{f_m(n) - f(n)\}\right| = \frac{1}{|u(n)|}\,|f_m(n) - f(n)|$$

$$\leq a\,|f_m(n) - f(n)| \to 0 \text{ as } m \to \infty \text{ for each } n \geq 1.$$

Hence $\dfrac{1}{u(n)} f_m(n)$ converges to $\dfrac{1}{u(n)} f(n)$ for each $n \geq 1$.

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Since $\left(\dfrac{f_m}{u}\right)/A_n$ is constant for each $n \geq 1$ and $f_m \to f$.

Thus $\left(\dfrac{f}{u}\right)/A_n$ is constant for each $n \geq 1$. Therefore, f belongs to $R\,(uc_\phi)$.

Hence $R\,(uc_\phi)$ is closed. ▪

Preposition 2**:** If $\lim_{n\to\infty} u(n) = 0$ then $R\,(uc_\phi)$ is closed if and only if $s\,(u)$ is finite.

Proof: **<u>Case-I</u>:** Suppose $s\,(u)$ is finite then $R\,(uc_\phi)$ is finite dimensional. Therefore $R\,(uc_\phi)$ is closed.

<u>Case-II</u>: Suppose $s\,(u)$ is an infinite set. Since $\lim_{n\to\infty} u(n) = 0$, it follows [24] that (uc_ϕ) is compact.

As set $s(u)$ is not finite, so range (uc_ϕ) is not finite dimensional. But a compact operator has closed range if and only if its range is finite dimensional. Hence $R(uc_\phi)$ is not closed.

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Preposition 3: If $\lim_{n\to\infty} u(n) \neq 0$, $s\,(u)$ is not finite and u is not bounded away from zero then $R\,(uc_\phi)$ is not closed.

Proof: Since u is not bounded away from zero there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that $u\,(n_k) \neq 0$; for each $k \geq 1$ and $u(n_k) \to 0$.

Let $v\,(n) = \begin{cases} 0 & \text{if } n \neq n_k \\ u(n_K) & \text{if } n = n_k \end{cases}$. Then $v \in l^\infty$ (as $u \in l^\infty$).

And $\lim_{k\to\infty} v(k) = 0$. Further $s\,(v)$ is not finite. Therefore $(v\,c_\phi)$ is not closed.

Hence there exists a function $f \in l^p$ such that $f \notin R\,(v\,c_\phi)$.

$f \notin R(u\,c_\phi) \Rightarrow f \neq v\,(g_o\phi)$ for each $g \in l^p$.

There exists a natural number k such that $f\,(k) \neq v\,(k)\,g\,(\phi(k))$

But $f \in \overline{R(v\;c_\phi)}$. Since $f \in \overline{R(v\;c_\phi)}$. Therefore $f\,(n) = 0$ whenever $n \notin s\,(v)$.

This implies that $f\,(n) = 0$ whenever $n \neq n_k$.

Let g be any vector in l^p then $f \notin R\,(v\,c_\phi)$.

This implies that $f \neq v\,(g_o\phi)$ $\Rightarrow f\,(n_k) \neq v\,(n_k)\,g\,(\phi(n_k))$

$\Rightarrow f(n_k) \neq u(n_k)\,g(\phi(n_k))$ for some $k \in$ **N.**

$\Rightarrow f \notin R(u\,c_\phi)$

But $f \in \overline{R(v\;c_\phi)} \Rightarrow f \in \overline{R(u\;c_\phi)}$. Therefore $R\,(u\,c_\phi)$ is not closed. ▪

Theorem1: Suppose $u \in l^{\infty}$ is a bounded away from zero. Then the range space of (uc_{ϕ}) is given by,

$R\,(uc_{\phi}) = \{f \in l^p : f / A_n$ is constant for each $n \in s(u)\}$ where $A_n = \{m \in s(u) : \phi(m) = n\}$.

Proof: Suppose f belongs to $R\,(uc_{\phi})$. For n in **N** and $A_n = \{m \in s(u) : \phi(m) = n\}$, we have to prove that f / A_n is constant. Let m_1 and m_2 be any two points in A_n. We need to show that $f(m_1) = f(m_2)$.

Since f belongs to $R\,(uc_{\phi})$, there is a function g in l^p for each $(uc_{\phi})(g) = f$.

In particular, $u(m_1)\, g(\phi(m_1)) = f(m_1) \Rightarrow g(\phi(m_1)) = \dfrac{f(m_1)}{u(m_1)}$

and $u(m_2)\, g(\phi(m_2)) = f(m_2) \Rightarrow g(\phi(m_2)) = \dfrac{f(m_2)}{u(m_2)}$

Given that m_1 and m_2 belongs to $s(u)$, therefore $u(m_1) \neq 0$ and $u(m_2) \neq 0$.

Since $\phi(m_1) = \phi(m_2) = n$, we get that

$g(n) = f(m_1) / u(m_1)$ and $g(n) = f(m_2) / u(m_2)$

Therefore $\dfrac{f(m_1)}{u(m_1)} = \dfrac{f(m_2)}{u(m_2)} = g(n)$

Therefore f / A_n is constant, for each $n \in s(u)$ where $A_n = \{m \in s(u) : \phi(m) = n\}$.

Conversely suppose that f belongs to l^p and f / A_n is constant for each n in **N.** Define g on **N** into **C** in the following way:

$$g(n) = \begin{cases} \dfrac{f(m_n)}{u(m_n)} & \text{for some } m \text{ belongs to } A_n \text{ if } A_n \text{ is not empty} \\ 0, & \text{if } A_n \text{ is empty.} \end{cases}$$

g is well defined because f is constant on A_n for each n. It can be easily seen that g belongs to l^p.

Also $(uc_{\phi})(g)(n) = u(n)\, g(\phi(n))$

$= f(n)$ for n belongs to **N.**

Thus $(uc_{\phi})(g) = f$. Hence f belongs to $R\,(uc_{\phi})$. ▪

Theorem 2: Suppose $u \in l^{\infty}$ is bounded away from zero. Then the weighted composition operator (uc_{ϕ}) on l^p is onto if and only if ϕ is one to one.

Proof: First suppose that ϕ is one to one. Let g be any function in l^p. Since ϕ is one to one, for each m in ϕ(**N**) there is unique n in **N** for each $\phi(n) = m$. Now we define a function f on **N** into **C** in the following way. For $m \in \phi$(**N**), let

$$f(\mathrm{m}) = \begin{cases} \dfrac{g(n)}{u(n)} & \text{where } n \text{ is the unique element of N such that } \phi(n) = m \\ 0, & \text{if } m \notin \phi(\mathrm{N}) \end{cases}$$

Now, $$\sum_{m \in \mathrm{N}} |f(m)|^2 = \sum_{m \in \phi(\mathrm{N})} |f(m)|^2 = \sum_{n \in \mathrm{N}} \left|\frac{g(n)}{u(n)}\right|^2 \leq \frac{1}{a^2} \sum |g(n)|^2 = \frac{1}{a^2} \|g\|^2$$

Hence f belongs to l^2.

We have $(uc_{\phi})(f)(n) = u(n)\, f(\phi(n)) = u(n)\, f(m) =$ $g(n)$

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Therefore $(uc_\phi)(f) = g$. Thus (uc_ϕ) is onto.

Conversely suppose that (uc_ϕ) is onto. We have to show that ϕ is one to one. For n_1 and n_2 in **N**, assume that $\phi(n_1) = \phi(n_2)$. The function $\chi_{n_1} \in l^p$. Since (uc_ϕ) is onto, there is a function g in l^p for each

$$(uc_\phi)(f) = \chi_{n_1}$$

Hence $(uc_\phi)(f)(n_1) = \chi_{n_1}(n_1) \Rightarrow u(n_1) f(\phi(n_1)) = 1$

$$\Rightarrow f(\phi(n_1)) = \frac{1}{u(n_1)} \quad (\because n_1 \in s(u))$$

Also $(uc_\phi)(f)(n_2) = \chi_{n_1}(n_2) = \delta_{n_1,n_2}$ where δ_{n_1,n_2} is kronecker delta. i.e.

$$u(n_2) f(\phi(n_2)) = \delta_{n_1,n_2} \Rightarrow f(\phi(n_2)) = \frac{\delta_{n_1,n_2}}{u(n_2)} \quad (\because n_2 \in s(u))$$

But, $\phi(n_1) = \phi(n_2)$. Thus $f(\phi(n_1)) = f(\phi(n_2))$.

Therefore $u(n_1) f(\phi(n_1)) = u(n_2) f(\phi(n_2))$.

Thus $1 = \delta_{n_1,n_2} \Rightarrow n_1 = n_2$. Hence ϕ is one to one. ▪

Theorem 3: Suppose $u \in l^\infty$ is bounded away from zero. Then the weighted composition operator (uc_ϕ) on l^p is one to one if and only if ϕ is onto.

Proof: Suppose that ϕ is onto. We need to show that uc_ϕ is one to one. For f *and* g in l^p, we have

$$(uc_\phi)(f) = (uc_\phi)(g) \Rightarrow (uc_\phi) f(n) = (uc_\phi) g(n) \text{ for each } n \text{ belongs to } \mathbf{N}.$$

$$\Rightarrow u(n) f(\phi(n)) = u(n) g(\phi(n)) \text{ for each } n \text{ belongs to } \mathbf{N}.$$

$$\Rightarrow f(\phi(n)) = g(\phi(n)) \text{ for each } n \text{ in } \mathbf{N}.$$

Thus $f = g$ (since ϕ is onto).

Therefore the weighted composition operator (uc_ϕ) is one to one.

Conversely suppose that (uc_ϕ) is one to one, we need to show that ϕ is onto.

Since $(uc_\phi)(0) = 0$ and (uc_ϕ) is one to one, it follows that for each natural number n.

$(uc_\phi)(\chi_n) \neq 0$ i.e. $u(n)\chi_{\phi^{-1}(n)} \neq 0 \Rightarrow \chi_{\phi^{-1}(n)} \neq 0$. Therefore ϕ-1(n) is non-empty for each natural number n. Hence ϕ is onto. ▪

Theorem 4: Suppose $u \in l^\infty$ is bounded away from zero and (uc_ϕ) is a weighted composition operator on l^p, then for $f = \sum f(n)\chi_n$

$$(uc_\phi)^*(f) = \sum \overline{u(n)} f(n) \chi_{\phi(n)} .$$

Proof: For each g in l^p, we have $\langle (uc_\phi)(g), \chi_n \rangle = \langle g, (uc_\phi)^* (\chi_n) \rangle$

In particular, the above equation is true for $g = \chi_m$ for each m in **N**.

Therefore $\langle (uc_\phi)(\chi_m), \chi_n \rangle = \langle \chi_m, (uc_\phi)^*(\chi_n) \rangle$. Now

$$\langle (uc_\phi)(\chi_m), \chi_n \rangle = (uc_\phi)(\chi_m)(n) = u(n)\, \chi_{\phi^{-1}(m)}(n) = \begin{cases} u(n) & \text{if } n \in \phi^{-1}(m) \\ 0, & \text{if } n \notin \phi^{-1}(m) \end{cases}$$

Also $\left\langle \chi_m, (uc_\phi)^*(\chi_n) \right\rangle = \overline{(uc_\phi)^*(\chi_n)(m)}$ where — denotes complex conjugation.

Thus $(uc_\phi)^*(\chi_n)(m) = \begin{cases} \overline{u(n)} & \text{if } n \in \phi^{-1}(m) \\ 0, & \text{if } n \notin \phi^{-1}(m) \end{cases}$

i.e. $(uc_\phi)^*(\chi_n)(m) = \begin{cases} \overline{u(n)} & \text{if } \phi(n) = m \\ 0, & \text{otherwise} \end{cases}$

i.e. $(uc_\phi)^*(\chi_n) = \overline{u(n)}\ \chi_{\phi(n)}$

If f = $\sum_{n=1}^{\infty} f(n)\chi_n$ then $(uc_\phi)^*(f) = \sum_{n=1}^{\infty} \overline{u(n)} f(n)\chi_{\phi(n)}$ ▪

Theorem 5: Suppose $u \in l^\infty$ is bounded away from zero and (uc_ϕ) be a weighted composition operator on l^p. Then the null space of $(uc_\phi)^*$ is given by,

$$N((uc_\phi)^*) = \left\{ f \in l^2 : \sum_{m \in A_n} f(m) = 0 \text{ for each } n \text{ in } s(u) \right\} \text{ where } A_n = \{m \in s(u) : \phi(m) = n\}.$$

Proof: Suppose that f belongs to N $((uc_\phi)^*)$, then $(uc_\phi)^*(f) = 0$. Assume that $f = \sum_{m \in \mathrm{N}} f(m)\chi_m$. Then $(uc_\phi)^*(f) = \sum_{m \in \mathrm{N}} \overline{u(m)} f(m)\chi_{\phi(m)}$ by theorem 4.

$$= \sum_{m \in \mathrm{N}} \left[\sum_{m \in A_n} \overline{u(m)} f(m)\chi_{\phi(m)} \right] = \sum_{m \in \phi(\mathrm{N})} \left[\sum_{m \in A_n} \overline{u(m)} f(m) \right] \chi_n$$

since $(uc_\phi)^*(f) = 0$, we get $\sum_{m \in A_n} \overline{u(m)} f(m) = 0$ for n in $\phi(\mathrm{N})$. But $\sum_{m \in A_n} \overline{u(m)} f(m) = 0$ for each n in $\phi(\mathrm{N})$. Thus we get $\sum_{m \in A_n} f(m) = 0$ for each $n \in s(u)$.

Conversely suppose that f belongs to l^p such that $\sum_{m \in A_n} f(m) = 0$ for each n in $s(u)$.

Then from the expression $(u\,c_\phi)^*(f) = \sum_{m \in \phi(\mathrm{N})} \left[\sum_{m \in A_n} \overline{u(m)} f(m) \right] \chi_n$

It follows easily that $(uc_\phi)^*(f) = 0$. Therefore f belongs to N $((uc_\phi)^*)$ ▪

Theorem 6: Suppose $u \in l^\infty$ is bounded away from zero and (uc_ϕ) be a weighted composition operator on l^p. Then the range space of $(uc_\phi)^*$ is given by,

$$R((uc_\phi)^*) = \left\{ f \in l^p : f / \mathrm{N} - \phi(\mathrm{N}) = 0 \right\}.$$

Proof: Suppose f belongs to R $((uc_\phi)^*)$. Then there exists a function g in l^p for each $(uc_\phi)^*(g) = f$. Suppose $g = \sum g(n)\chi_n$. Then $(uc_\phi)^*(g) = \sum \overline{u(n)}\ g(n)\ \chi_{\phi(n)}$

It follows that for m belongs to $\mathrm{N} - \phi(\mathbf{N})$, $(uc_\phi)^*(g)(m) = 0$

Therefore f $(m) = 0$ for each $m \in \mathrm{N} - \phi(\mathbf{N})$

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Conversely assume that $f \in l^p$ and $f(m) = 0$ for each $m \in \mathrm{N} - \phi(\mathbf{N})$. We need to show that $f \in$ R $((uc_\phi)^*)$. We have $\mathrm{N} = \bigcup_{n\in\phi(\mathrm{N})} A_n$, where $A_n = \{m \in \mathrm{N} : \phi(m) = n\}$.

We notice that for $n \in \phi(\mathbf{N})$, each A_n is non empty finite subset of **N**. Let $\overline{\overline{A}}_n$ denote the number of elements in A_n. Let

$$g = \sum_{\substack{m=1\\ \phi(m)=n}}^{\infty} \frac{f(n)}{\overline{u(n)}\ \overline{\overline{A}}_n} \chi_m$$

Then $$\sum_{m\in \mathrm{N}} |g(m)|^2 = \sum_{n\in\phi(\mathrm{N})}\left[\sum_{m\in A_n}|g(m)|^2\right] = \sum_{n\in\phi(\mathrm{N})}\left[\sum_{m\in A_n}\left|\frac{f(n)}{\overline{u(n)}\ \overline{\overline{A}}_n}\right|^2\right]$$

$$= \sum_{n\in\phi(\mathrm{N})}\left[\overline{\overline{A}}_n \frac{|f(n)|^2}{\left|\overline{u(n)}\right|^2 \overline{\overline{A}}_n^{\,2}}\right] = \sum_{n\in\phi(\mathrm{N})} \frac{|\ f(n)\ |^2}{\left|\overline{u(n)}\right|^2 \overline{\overline{A}}_n}$$

$$\leq \sum_{n\in\phi(\mathrm{N})} \frac{|f(n)|^2}{\left|\overline{u(n)}\right|^2} < \infty$$ (since $\overline{\overline{A}}_n \geq 1$ for each $n \in \phi(\mathbf{N})$ and $u \in l^\infty$)

Hence g belongs to l^p. Now we shall show that $(uc_\phi)^*(g) = f$. We have,

$$(uc_\phi)^*(g) = \sum_{\substack{m=1\\ \phi(m)=n}}^{\infty} \frac{\overline{u(n)}\ f(n)}{\overline{u(n)}\ \overline{\overline{A}}_n} \chi_{\phi(m)} = \sum_{n\in\phi(\mathrm{N})}\left[\sum_{m\in A_n}\frac{f(n)}{\overline{\overline{A}}_n}\chi_{\phi(m)}\right]$$

$$= \sum_{n\in\phi(\mathrm{N})}\left[\overline{\overline{A}}_n \frac{f(n)}{\overline{\overline{A}}_n}\chi_{(n)}\right] = \sum_{n\in\phi(\mathrm{N})} f(n)\chi_{(n)} = f\,(\text{since } f\,/\,\mathrm{N}-\phi(\mathrm{N})=0) \quad ▪$$

Corollary: The range space R$((uc_\phi)^*)$ is a closed subspace of l^p.

Proof: Let $(f_n)_{n=1}^{\infty}$ be a sequence in R$((uc_\phi)^*)$ for each $(f_n)_{n=1}^{\infty}$ converges to f in l^p. Since $f_n / \mathrm{N}-\phi(\mathrm{N}) = 0$ and converges in l^p implies pointwise convergence, it follows that the limit function $f = 0$ on $\mathrm{N}-\phi(\mathrm{N})$. Thus $f \in$ R$((uc_\phi)^*)$. Hence R$((uc_\phi)^*)$ is a closed subspace of l^p.

Theorem 7: Suppose $u \in l^\infty$ is bounded away from zero and (uc_ϕ) be a weighted composition operator on l^p. Then the adjoint $(uc_\phi)^*$ of weighted composition operator (uc_ϕ) is one to one if and only if ϕ is one to one.

Proof: Suppose $(uc_\phi)^*$ is a one to one. We need to show that ϕ is one to one. For n and m in **N**. $\phi(n) = \phi(m)$ $\Rightarrow \chi_{\phi(n)} = \chi_{\phi(m)} \Rightarrow (uc_\phi)^*(\chi_n) = (uc_\phi)^*(\chi_m) \Rightarrow \chi_n = \chi_m$ (since $(uc_\phi)^*$ is one to one) $\Rightarrow$ $n = m$ $\Rightarrow \phi$ is one to one.

Conversely assume that ϕ is one to one. We need to show that $(uc_\phi)^*$ is one to one.

For $f = \sum_{n=1}^{\infty} f(n)\chi_n$ and $g = \sum_{n=1}^{\infty} g(n)\chi_n$ in l^p, $(uc_\phi)^*$ (f) = $(uc_\phi)^*$ (g). This implies that $\sum_{n=1}^{\infty}\overline{u(n)}f(n)\chi_{\phi(n)} = \sum_{n=1}^{\infty}\overline{u(n)}g(n)\chi_{\phi(n)}$. Thus $f(n) = g(n)$ for each $n \in \mathbf{N}$ because ϕ is one to one. Therefore f = g. Hence $(uc_\phi)^*$ is one to one. ▪

Theorem 8: Suppose $u \in l^\infty$ is bounded away from zero and (uc_ϕ) is a weighted composition operator on l^p. Then the adjoint $(uc_\phi)^*$ of a weighted composition operator (uc_ϕ) is onto if and only if ϕ is onto.

Proof: Suppose $(uc_\phi)^*$ is onto. We need to show that ϕ is onto. Let m be any natural number, then $\chi_m \in l^p$. Since $(uc_\phi)^*$ is onto, there is a function $f = \sum f(n)\chi_n$ in l^p such that

$(uc_\phi)^*$ $(f) = \chi_m$ i.e. $\sum\overline{u(n)}f(n)\chi_{\phi(n)} = \chi_m$.

There is a natural number n for each $\phi(n) = m$. Hence ϕ is onto.

Conversely assume that ϕ is onto. We need to show that $(uc_\phi)^*$ is onto. Let $g = \sum_{m=1}^{\infty} g(m)\chi_m$ be any function in l^p. Since ϕ is onto for each m in $\mathbf{N}$, $A_m = \phi^{-1}(m)$ is non-empty. Let P be a set consisting of precisely one number from each of the sets A_n for m in $\mathbf{N}$. Let f = $\sum_{\substack{m\in N \\ n\in P,\phi(n)=m}} \frac{g(m)}{u(m)}\chi_n$. Then

$$(uc_\phi)^*(f) = \sum_{\substack{m\in N \\ n\in P,\phi(n)=m}} \overline{u(m)}\,\frac{g(m)}{\overline{u(m)}}\chi_{\phi(n)} = \sum_{m\in N} g(m)\chi_m = g.$$

Hence $(uc_\phi)^*$ is onto. ▪

Corollary: Suppose $u \in l^\infty$ is bounded away from zero then $(uc_\phi)^*$ is invertible if and only if ϕ is invertible.

Proof: Proof follows from theorem 7 and 8.

Theorem 9: Suppose $u \in l^\infty$ is bounded away from zero. Then the adjoint $(uc_\phi)^*$ of a weighted composition operator (uc_ϕ) is a weighted composition operator if and only if ϕ is one to one and onto.

Proof: Suppose ϕ is one to one and onto. We need to show that adjoint $(uc_\phi)^*$ is a Weighted composition operator. We have to find a function ψ on $\mathbf{N}$ into itself such that

$$(uc_\phi)^*(f) = \bar{u}\, c_\psi(f) \text{ for each } f \text{ in } l^p.$$

Let $\psi = \phi^{-1}$ (since ϕ is one to one and onto). To show that $(uc_\phi)^* = (\bar{u}\, c_\psi)$, it is sufficient to show that $(uc_\phi)^*$ $(\chi_n) = (u'c_\psi)$ (χ_n) for each $n \in s(u)$.

Thus we need to show that $(uc_\phi)^*$ (χ_n) $(m) = (u'c_\psi)$ (χ_n) (m).

i.e. $\overline{u(m)}\,\chi_{\phi(n)}(m) = u'(m)\,\chi_{\psi^{-1}(n)}(m)$

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But $\psi^{1}(n) = (\phi^{-1})^{-1}(n) = \phi(n)$. Thus we have to show that $\overline{u(m)}\,\chi_{\phi(n)}(m) = \overline{u'(m)}\,\chi_{\phi(n)}(m)$ which is obviously true. Therefore $(uc_\phi)^*$ is a weighted composition operator and $(uc_\phi)^* = \bar{u}c_{\phi^{-1}}$.

Conversely assume that $(uc_\phi)^*$ is a weighted composition operator. We need to show that ϕ is one to one and onto and $(uc_\phi)^* = \bar{u}c_{\phi^{-1}}$. Since $(uc_\phi)^*$ is a weighted composition operator there is a function ψ on **N** into itself for each $(uc_\phi)^* = u'c_\psi$. Therefore $(uc_\phi)^*(\chi_n) = (u'c_\psi)(\chi_n)$ for each $n \in s(u)$. Thus $(uc_\phi)^*(\chi_n)(m) = (u'c_\psi)(\chi_n)(m)$ for each $m \in \mathbf{N}$.

Therefore $\overline{u(m)}\,\chi_{\phi^{-1}(n)}(m) = u'(m)\,\chi_{\psi^{-1}(n)}(m)$.

Hence $\overline{u(m)}\,\chi_{\phi^{-1}(n)}(m) = u'(m)\chi_n(\psi(m))$.. ... (1)

Let $m = \phi(n)$ in eqn. (1), then we get $\overline{u(m)}\,\chi_{\phi(n)}(\phi(n)) = u'(m)\chi_n(\psi(\phi(n)))$.

Therefore $\psi(\phi(n)) = n$, and $u'(n) = \overline{u(n)}$ for each $n \in s(u)$. Thus $\psi \circ \phi = I$.

Hence ϕ is one to one.

Again in the equation (1) we put $n = \psi(m)$, we get $\chi_{\phi(\psi(m))}(m) = \chi_{\psi(m)}(\psi(m))$

Thus we get $\phi(\psi(m)) = m$. Therefore $\phi \circ \psi = I$. Hence ϕ is onto.

Thus ϕ is one to one and onto. ▪

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